

NORMAL HILBERT POLYNOMIALS : A SURVEY

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ABSTRACT. We survey some of the major results about normal Hilbert polynomials of ideals. We discuss a formula due to Lipman for complete ideals in regular local rings of dimension two, theorems of Huneke, Itoh, Huckaba, Marley and Rees in Cohen-Macaulay analytically unramified local rings. We also discuss recent works of Goto-Hong-Mandal and Mandal-Singh-Verma concerning the positivity of the first coefficient of the normal Hilbert polynomial in unmixed analytically unramified local rings. Results of Moralés and Villarreal linking normal Hilbert polynomial of monomial ideals with Ehrhart polynomials of polytopes are also presented.

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INTRODUCTION

In this paper we survey results about normal Hilbert polynomials of ideals in local Noetherian rings and polynomial rings. We first set-up the notation and then describe the contents of this paper. Let I be an \mathfrak{m} -primary ideal of a local ring (R, \mathfrak{m}) of dimension d . A sequence of ideals $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$ is

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called an **I -admissible filtration** if there exists a $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{Z}$,

$$(i) \ I_{n+1} \subseteq I_n, (ii) \ I_m I_n \subseteq I_{m+n} \text{ and } (iii) \ I^n \subseteq I_n \subseteq I^{n-k}.$$

Marley in [27] showed that if \mathcal{I} is an I -admissible filtration then the **Hilbert function** of \mathcal{I} defined by $H_{\mathcal{I}}(n) = \lambda(R/I_n)$ where λ denotes length as R -module coincides with a polynomial $P_{\mathcal{I}}(x) \in \mathbb{Q}[x]$ of degree d for large n . This polynomial is written as

$$P_{\mathcal{I}}(x) = e_0(\mathcal{I}) \binom{x+d-1}{d} - e_1(\mathcal{I}) \binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{I})$$

and it is called the **Hilbert polynomial of \mathcal{I}** . The uniquely determined integers $e_i(\mathcal{I})$ for $i = 0, 1, \dots, d$ are called the **Hilbert coefficients of \mathcal{I}** . The coefficient $e_0(\mathcal{I})$ is a positive integer and it is called the **multiplicity of \mathcal{I}** . The coefficient $e_1(\mathcal{I})$ is called the **Chern number of \mathcal{I}** . If \mathcal{I} is the I -adic filtration, then $e(I) := e_0(\mathcal{I}) = e_0(I)$ (resp. $e_1(\mathcal{I}) := e_1(I)$) is called the **multiplicity** (resp. the **Chern number**) of I .

We set

$$n(\mathcal{I}) = \sup\{n \in \mathbb{Z} \mid H_{\mathcal{I}}(n) \neq P_{\mathcal{I}}(n)\}.$$

The integer $n(\mathcal{I})$ is called the **postulation number** of \mathcal{I} . For an ideal I in a ring R , the **integral closure** of I , denoted by \bar{I} , is the set of elements $x \in R$ such that x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

where $a_j \in I^j$ for $1 \leq j \leq n$. If $x \in \bar{I}$ we say x is integral over I . Note that \bar{I} is an ideal. An ideal I is said to be **integrally closed** or **complete** if $\bar{I} = I$ and it is said to be **normal** if all its powers are integrally closed. An ideal I is said to be **asymptotically normal** if there exists an integer $N \geq 1$ such that I^n is integrally closed for all $n \geq N$. A ring R is said to be **analytically unramified** if its \mathfrak{m} -adic completion \hat{R} is reduced. Rees in [33] showed that if R is analytically unramified then the integral closure filtration $\{\bar{I}^n\}$ is an I -admissible filtration. It follows that if I is an \mathfrak{m} -primary ideal then the **normal Hilbert function** $\bar{H}_I(n) = \lambda(R/\bar{I}^n)$ coincides with a polynomial $\bar{P}_I(n)$ of degree d for large n . This polynomial is referred to as the **normal Hilbert polynomial** of I and it is written in the form

$$\bar{P}_I(x) = \bar{e}_0(I) \binom{x+d-1}{d} - \bar{e}_1(I) \binom{x+d-2}{d-1} + \cdots + (-1)^d \bar{e}_d(I)$$

where $\bar{e}_0(I), \dots, \bar{e}_d(I)$ are integers uniquely determined by I . These integers are known as **normal Hilbert coefficients** of I . The integer $\bar{e}_1(I)$ is called the **normal Chern number of I** . The Hilbert series of an I -admissible filtration $\mathcal{I} = \{I_n\}$, is defined as

$$F_{\mathcal{I}}(t) = \sum_{n \geq 1} \lambda(I_{n-1}/I_n) t^{n-1}.$$

Given any filtration $\mathcal{I} = \{I_n\}$, the Rees algebra, extended Rees algebra and the associated graded ring of \mathcal{I} are defined as

$$\mathcal{R}_+(\mathcal{I}) = \bigoplus_{n \geq 0} I_n t^n, \quad \mathcal{R}(\mathcal{I}) = \bigoplus_{n \in \mathbb{Z}} I_n t^n \quad \text{and} \quad G(\mathcal{I}) = \bigoplus_{n \geq 0} I_n / I_{n+1}$$

respectively. For the integral closure filtration $\mathcal{I} = \{\overline{I^n}\}$ we denote the Hilbert series $F_{\mathcal{I}}(t)$ by $\overline{F}_I(t)$ the Rees algebra by $\overline{\mathcal{R}}_+(I)$, extended Rees algebra by $\overline{\mathcal{R}}(I)$ and the associated graded ring by $\overline{G}(I)$.

A **reduction** of an I -admissible filtration $\mathcal{I} = \{I_n\}$ is an ideal $J \subseteq I_1$ such that $J I_n = I_{n+1}$ for all large n . Equivalently $J \subseteq I_1$ is a reduction of \mathcal{I} if and only if $\mathcal{R}(\mathcal{I})$ is a finite $\mathcal{R}(J)$ -module. A **minimal reduction** of \mathcal{I} is a reduction of \mathcal{I} minimal with respect to inclusion. For a minimal reduction J of \mathcal{I} , we set

$$r_J(\mathcal{I}) = \sup\{n \in \mathbb{Z} \mid I_n \neq J I_{n-1}\}.$$

The **reduction number** $r(\mathcal{I})$ of \mathcal{I} is defined to be the least $r_J(\mathcal{I})$ over all possible minimal reductions J of \mathcal{I} . If $\mathcal{I} = \{\overline{I^n}\}$ then we write $r(\mathcal{I}) := \bar{r}(I)$. The coefficient $e_0(\mathcal{I})$ has been studied extensively. We shall not discuss it in this paper. Our purpose here is to focus on the other coefficients about which not much is known.

In the first section, we discuss results about $e_1(\mathcal{I})$ for an I -admissible filtration. The principal result is a theorem of Huckaba and Marley which gives sharp lower and upper bounds for this coefficient and relates them to the depth of $G(\mathcal{I})$. Using this result, we discuss when $e_1(\mathcal{I})$ vanishes. We will also discuss conditions under which the normal reduction number is at most one or at most two. Use of Huckaba-Marley Theorem simplifies the proofs of these results.

In the second section, we discuss a remarkable result due to Moralés, Trung and Villamayor. It answers the question: when is $e_1(I) = \bar{e}_1(I)$ for a parameter ideal in an analytically unramified excellent local domain? Their result states that it is possible only when I is normal and R is regular. We will see that this is true for unmixed analytically unramified local rings.

In the third section we continue the study of the normal Chern number. In this section we sketch the recent solution of the Positivity Conjecture of Vasconcelos due to Goto, Hong and Mandal [10].

We shall survey main results about the second Hilbert coefficient in the fourth section. The most decisive result in this direction is due to Marley. It states that in a Cohen-Macaulay local ring $e_2(\mathcal{I}) \geq 0$ for any I -admissible filtration \mathcal{I} . We discuss this for $\bar{e}_2(I)$ using a formula of Blancafort for the difference of Hilbert polynomial and Hilbert function of an I -admissible filtration in terms of lengths of local cohomology modules of the Rees algebra $\mathcal{R}(\mathcal{I})$ with support in $\bigoplus_{n>0} I_n t^n$. We shall give a new proof of Itoh's lower bound $\bar{e}_2(I) \geq \bar{e}_1(I) - \lambda(\bar{I}/I)$ and show that equality holds if and only if $\bar{r}(I) \leq 2$. And in this case $\bar{G}(I)$ is Cohen-Macaulay.

In the fifth section, we study the third normal Hilbert coefficient. This coefficient is perhaps the most interesting. Marley constructed an example of a monomial ideal I in the power series ring $k[[x, y, z]]$ for which $e_3(I) = -1$. In contrast to this, Itoh proved that if R is any analytically unramified Cohen-Macaulay local ring then $\bar{e}_3(I) \geq 0$. This is proved by using a local cohomological interpretation of this coefficient. One has to invoke a special case of Itoh's vanishing theorem : $[H_{\mathcal{R}(I)_+}^2 \bar{\mathcal{R}}(I)]_n = 0$ for all $n \leq 0$ if $\dim R \geq 3$. Itoh proved that if $\bar{r}(I) \leq 2$ and R is Cohen-Macaulay analytically unramified local ring of dimension at least 3 then $\bar{e}_3(I) = 0$.

In fact he showed that in this case $\bar{G}(I)$ is Cohen-Macaulay. Itoh conjectured that if R is Gorenstein local ring of dimension at least three and $\bar{e}_3(I) = 0$ then $\bar{r}(I) \leq 2$. We will present his solution of this conjecture for the filtration $\{\bar{I}^n\}$ for parameter ideals I for which $\bar{I} = \mathfrak{m}$.

The normal Hilbert polynomial of an \mathfrak{m} -primary ideal in a two-dimensional regular local ring (R, \mathfrak{m}) was computed by Rees and Lipman. In the sixth section, we present a proof using Zariski's theory of complete ideals for computation of the normal Hilbert function $\lambda(R/(\bar{I}^r J^s))$ for \mathfrak{m} -primary ideals I and J . The formula of Rees and Lipman is a special case of it. This formula also follows from a formula of Hoskin and Deligne for $\lambda(R/I)$ where I is a complete \mathfrak{m} -primary ideal of R .

Finally in section seven, we study the normal Hilbert polynomial of a zero-dimensional monomial ideal in the polynomial ring $R = k[x_1, x_2, \dots, x_n]$ over a field k . In this case, this polynomial is expressed as a difference of two Ehrhart polynomials derived from the exponent vectors of the monomials generating I .

1. STUDY OF $e_1(\mathcal{I})$

In this section we discuss results about the first normal Hilbert coefficient $\bar{e}_1(I)$. The first general result was proved by T. Marley in [27].

Theorem 1.1. [27, Lemma 3.19] *Let (R, \mathfrak{m}) be a local ring and let I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Then*

$$e_1(\mathcal{I}) \geq e_1(I).$$

In particular if R is an analytically unramified Cohen-Macaulay local ring then $\bar{e}_1(I) \geq 0$.

Proof. We may assume the residue field R/\mathfrak{m} is infinite and then by using superficial elements [18, section 8.5] we can reduce to the case $\dim R = 1$. Since $I^n \subseteq I_n$ for all n , $P_{\mathcal{I}}(n) \leq P_I(n)$ for n sufficiently large. Then

$$e_0(\mathcal{I})n - e_1(\mathcal{I}) \leq e_0(I)n - e_1(I)$$

for n sufficiently large. Since $e_0(\mathcal{I}) = e_0(I)$, so $e_1(\mathcal{I}) \geq e_1(I)$. Now let R be an analytically unramified Cohen-Macaulay local ring. Let J be a minimal reduction of I . Then $\overline{I^n} = \overline{J^n}$ for all n . Thus

$$\bar{e}_1(I) = \bar{e}_1(J) \geq e_1(J) = 0.$$

□

Huckaba and Marley [13] have given lower and upper bounds for the Chern number of an I -admissible filtration in a Cohen-Macaulay local ring.

Theorem 1.2 (Huckaba-Marley Theorem). [13, Theorem 4.7] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and let I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration and J be a minimal reduction of \mathcal{I} . Then*

- (1) $\sum_{n \geq 1} \lambda((I_n, J)/J) \leq e_1(\mathcal{I}) \leq \sum_{n \geq 1} \lambda(I_n/JI_{n-1}).$
- (2) $e_1(\mathcal{I}) = \sum_{n \geq 1} \lambda((I_n, J)/J)$ if and only if $G(\mathcal{I})$ is Cohen-Macaulay.
- (3) $e_1(\mathcal{I}) = \sum_{n \geq 1} \lambda(I_n/JI_{n-1})$ if and only if $\text{depth } G(\mathcal{I}) \geq d - 1$.

This theorem has been proved for modules in [36, Theorem 2.5, 2.7]. Before we describe consequences of the Huckaba-Marley Theorem, we state the Valabrega-Valla criterion for Cohen-Macaulayness of $G(\mathcal{I})$ for an I -admissible filtration \mathcal{I} . This plays a fundamental role in the study of Hilbert polynomials.

Theorem 1.3 (Valabrega-Valla, [39]). *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and let I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Let $J = (x_1, \dots, x_d)$ be a minimal reduction of \mathcal{I} and $\underline{x}^* = x_1^*, \dots, x_d^*$ be their images in I_1/I_2 . Then \underline{x}^* is a regular sequence if and only if $J \cap I_n = JI_{n-1}$ for all $n \geq 1$.*

The next result of Marley shows the consequences of the vanishing of the Chern number of an I -admissible filtration in Cohen-Macaulay local ring.

Corollary 1.4. [27, Theorem 3.21] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Then the following are equivalent:*

- (1) I is generated by a system of parameters and $\mathcal{I} = \{I^n\}$,
- (2) $e_1(\mathcal{I}) = \dots = e_d(\mathcal{I}) = 0$,
- (3) $e_1(\mathcal{I}) = 0$.

Proof. (1) \Rightarrow (2) : Since R is Cohen-Macaulay, I is generated by a regular sequence, $G(I) \simeq R/I[x_1, x_2, \dots, x_d]$ where x_1, x_2, \dots, x_d are indeterminates. Therefore for all $n \geq 1$,

$$P_I(n) = \lambda(R/I) \binom{n+d-1}{d}.$$

Hence $e_1(I) = e_2(I) = \dots = e_d(I) = 0$.

(2) \Rightarrow (3) : This is clear.

(3) \Rightarrow (1) : Let $e_1(\mathcal{I}) = 0$. Then by Theorem 1.2, $G(\mathcal{I})$ is Cohen-Macaulay and $I_n \subset J$ for all n . By Theorem 1.3, $I_n \cap J = JI_{n-1} = I_n$ for all $n \geq 1$. This gives $I_n = J^n$ for all $n \geq 1$. \square

As a consequence of the above result in an analytically unramified Cohen-Macaulay local ring if $\bar{e}_1(I) = 0$ then $\bar{I}^n = I^n$ for all n , $\bar{e}_j(I) = 0$ for $j \geq 1$ and $\bar{r}(I) = 0$.

Corollary 1.5. [13, Corollary 4.9] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with R/\mathfrak{m} infinite and let I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Then $\lambda(R/I_1) \geq e_0(\mathcal{I}) - e_1(\mathcal{I})$ and equality holds if and only if $r(\mathcal{I}) \leq 1$.*

Proof. Let J be a minimal reduction of \mathcal{I} . Since

$$\lambda((I_1, J)/J) \leq \sum_{n \geq 1} \lambda((I_n, J)/J) \leq e_1(\mathcal{I})$$

we have

$$\lambda(I_1/J) = e_0(\mathcal{I}) - \lambda(R/I_1) \leq e_1(\mathcal{I}).$$

If $\lambda(I_1/J) = e_0(\mathcal{I}) - \lambda(R/I_1) = e_1(\mathcal{I})$, we get $\lambda((I_n, J)/J) = 0$ for all $n \geq 2$. By Theorem 1.3, $I_n = JI_{n-1}$ for all $n \geq 2$. Hence $r(\mathcal{I}) \leq 1$. Conversely let $r(\mathcal{I}) \leq 1$. Then by Theorem 1.2, $\lambda(I_1/J) \leq e_1(\mathcal{I}) \leq \lambda(I_1/J)$. Hence $\lambda(I_1/J) = e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1)$. \square

The next result of Huckaba-Marley characterizes Cohen-Macaulay property of the Rees algebra of an I -admissible filtration in terms of its Chern number.

Corollary 1.6. [13, Corollary 4.10] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and I be an \mathfrak{m} -primary ideal. Let R/\mathfrak{m} be infinite. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration and J be a minimal reduction of \mathcal{I} . Then $\mathcal{R}_+(\mathcal{I})$ is Cohen-Macaulay if and only if $e_1(\mathcal{I}) = \sum_{n=1}^{d-1} \lambda((I_n, J)/J)$.*

Proof. We first prove that $e_1(\mathcal{I}) = \sum_{n=1}^{d-1} \lambda((I_n, J)/J)$ if and only if $G(\mathcal{I})$ is Cohen-Macaulay and $r(\mathcal{I}) < d$. The if part follows from Theorem 1.2 (1). Now suppose the equality holds. By Theorem 1.2(1), $G(\mathcal{I})$ is Cohen-Macaulay and $I_n \subseteq J$ for all $n \geq d$. By Theorem 1.3, $J \cap I_n = JI_{n-1}$ for all $n \geq 1$. Thus $I_n = JI_{n-1}$ for all $n \geq d$. The rest follows from [42]. \square

The next result describes the relationship between the postulation number and reduction number of an I -admissible filtration.

Theorem 1.7. [27, Corollary 3.8] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with an infinite residue field. Let I be an \mathfrak{m} -primary ideal and $\mathcal{I} = \{I_n\}$ be an I -admissible filtration such that $\text{depth } G(\mathcal{I}) \geq d-1$. Then $r(\mathcal{I}) = n(\mathcal{I}) + d$.*

Itoh in [20] has given another lower bound for the normal Chern number in a Cohen-Macaulay local ring. We give a different proof of Itoh's theorem. Before we prove it we need some more results of Itoh proved in [19].

Theorem 1.8 (Huneke-Itoh Intersection Theorem). [19, Theorem 1] *Let (R, \mathfrak{m}) be a d -dimensional local ring and I be an ideal generated by a regular sequence. Then for every $n \geq 1$,*

$$I^n \cap \overline{I^{n+1}} = I^n \overline{I}.$$

Recently Ulrich and Hong have given a simpler proof in a more general settings of the Huneke-Itoh Intersection Theorem [11].

Lemma 1.9. *Let (R, \mathfrak{m}) be a local ring and I be a parameter ideal of R . Then $\bar{r}(I) \leq 2$ if and only if $\overline{I^{n+2}} = I^n \overline{I^2}$ for all $n \geq 1$.*

Proof. We apply induction on n . If $\bar{r}(I) \leq 2$, $\overline{I^{n+2}} = I \overline{I^{n+1}}$ for all $n \geq 1$. Hence $\overline{I^3} = I \overline{I^2}$. Thus the result holds for $n = 1$. Now assume the results holds for n and we will prove it for $n+1$. Note that $\overline{I^{n+3}} = I \overline{I^{n+2}} = I^{n+1} \overline{I^2}$. Conversely assume that $\overline{I^{n+2}} = I^n \overline{I^2}$ for all $n \geq 1$. Apply induction on n . Then $\overline{I^{n+3}} = I^{n+1} \overline{I^2} = I \overline{I^{n+2}}$. Hence $\bar{r}(I) \leq 2$. \square

Next we prove a result of Itoh about a lower bound on the normal Chern number. For a generalisation of this result to good filtration of modules, see [36, Theorem 3.1].

Theorem 1.10. [20, Theorem 2(1)] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified Cohen-Macaulay local ring and let I be a parameter ideal. Then*

$$\bar{e}_1(I) \geq \lambda(\bar{I}/I) + \lambda(\overline{I^2}/I\bar{I})$$

and equality holds if and only if $\bar{r}(I) \leq 2$.

Proof. Applying Theorem 1.2,

$$\begin{aligned} \bar{e}_1(I) &\geq \lambda(\bar{I}/I) + \lambda((\overline{I^2}, I)/I) \\ &= \lambda(\bar{I}/I) + \lambda(\overline{I^2}/(\overline{I^2} \cap I)) \\ &= \lambda(\bar{I}/I) + \lambda(\overline{I^2}/I\bar{I}) \text{ (by Huneke-Itoh intersection theorem).} \end{aligned}$$

Suppose that equality holds, i.e. $\bar{e}_1(I) = \lambda(\bar{I}/I) + \lambda(\overline{I^2}/I\bar{I})$. By Theorem 1.2, $\lambda((\overline{I^n}, I)/I) = 0$ for all $n \geq 3$, which implies $\overline{I^n} \subseteq I$ for $n \geq 3$. By Theorem 1.2, $\bar{G}(I)$ is Cohen-Macaulay. Hence by Valabrega-Valla theorem we have $\overline{I^n} \cap I = I \overline{I^{n-1}}$ for all $n \geq 1$. Thus we have $\overline{I^{n+2}} = \overline{I^{n+2}} \cap I = I \overline{I^{n+1}}$ for all $n \geq 1$. Hence $\bar{r}(I) \leq 2$.

Conversely assume that $\bar{r}(I) \leq 2$. Then we have $\overline{I^{n+2}} = I \overline{I^{n+1}}$ for all $n \geq 1$. By Theorem 1.2(1), we get $\bar{e}_1(I) \leq \lambda(\bar{I}/I) + \lambda(\overline{I^2}/I\bar{I})$. Hence we have

$$\bar{e}_1(I) - \lambda(\bar{I}/I) = \lambda(\overline{I^2}/I\bar{I}).$$

\square

2. ON THE EQUALITY $\bar{e}_1(I) = e_1(I)$

In this section we observe how the equality of the Chern number and normal Chern number characterizes the ring to be regular. In this direction Moralés, Trung and Villamayor proved the following interesting

Theorem 2.1. [30, Theorem 1.2] *Let (R, \mathfrak{m}) be an analytically unramified excellent local domain and let I be a parameter ideal. If $\bar{e}_1(I) = e_1(I)$ then R is regular and $\overline{I^n} = I^n$ for all n .*

We prove that this result is true for analytically unramified unmixed local domains. A local ring R is called **unmixed** if $\dim \hat{R}/\mathcal{P} = \dim R$ for each associated prime ideal \mathcal{P} of \hat{R} . In order to prove the theorem we need to recall the theory of \mathfrak{m} -full ideals introduced by Rees. First we present Goto's Theorem regarding the characterization of regular local rings in terms of \mathfrak{m} -full ideals. We say that an ideal I is **\mathfrak{m} -full**, if $\mathfrak{m}I : x = I$ for some $x \in \mathfrak{m}$.

Lemma 2.2. [9, Lemma 2.2] *Let (R, \mathfrak{m}) be a d -dimensional local ring and I be an \mathfrak{m} -primary ideal. Then*

(1) $\mu(I) \leq \lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \lambda(R/I + xR) + \mu(I + xR/xR)$ for any element x of \mathfrak{m} .

(2) If I is \mathfrak{m} -full then $\mu(J) \leq \mu(I)$ for any ideal J of R such that $I \subseteq J$.

Proof. (1) Consider the exact sequence

$$0 \longrightarrow \mathfrak{m}I : x/\mathfrak{m}I \longrightarrow R/\mathfrak{m}I \xrightarrow{x} R/\mathfrak{m}I \longrightarrow R/\mathfrak{m}I + xR \longrightarrow 0.$$

It follows from the above sequence that $\lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \lambda(R/\mathfrak{m}I + xR)$. Since $I \subseteq \mathfrak{m}I : x$, we have $\mu(I) = \lambda(I/\mathfrak{m}I) \leq \lambda(\mathfrak{m}I : x/\mathfrak{m}I)$. The equality follows from

$$\begin{aligned} \lambda(R/(I, x)) + \mu((I, x)/(x)) &= \lambda(R/(I, x)) + \lambda(I + xR/\mathfrak{m}I + xR) \\ &= \lambda(R/(\mathfrak{m}I, xR)). \end{aligned}$$

(2) Since I is \mathfrak{m} -full there exists $x \in \mathfrak{m}$ such that $\mathfrak{m}I : x = I$. Hence $\mu(J) \leq \lambda(J/\mathfrak{m}I + xJ)$, as $\mathfrak{m}I + xJ \subseteq \mathfrak{m}J$. From the exact sequence given below

$$0 \longrightarrow I/\mathfrak{m}I \longrightarrow J/\mathfrak{m}I \xrightarrow{x} J/\mathfrak{m}I \longrightarrow J/\mathfrak{m}I + xJ \longrightarrow 0$$

it follows that $\mu(I) = \lambda(J/\mathfrak{m}I + xJ)$. Thus we have $\mu(J) \leq \mu(I)$. \square

Proposition 2.3. [9, Proposition 2.3] *Let (R, \mathfrak{m}) be a local ring of dimension $d \geq 1$ and I be a parameter ideal. Then I is \mathfrak{m} -full if and only if R is regular and $\lambda(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 1$.*

Proof. Let I be \mathfrak{m} -full. Let $x \in \mathfrak{m}$ such that $\mathfrak{m}I : x = I$. Then by Lemma 2.2, we have $\mu(\mathfrak{m}) \leq \mu(I) = d$. Thus R is regular. Note that by Lemma 2.2, we have

$$\lambda(R/I + xR) + \mu(I + xR/xR) = \mu(I) = d.$$

But $\lambda(R/I + xR) \geq 1$ and $\mu(I + xR/xR) \geq d - 1$. Hence we must have $I + xR = \mathfrak{m}$. Thus $\lambda(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 1$.

Conversely assume that R is regular and $\lambda(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 1$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m} = I + xR$. Then we have $\lambda(\mathfrak{m}I : x/\mathfrak{m}I) = d$ by Lemma 2.2. We also have $\mu(I) = \lambda(I/\mathfrak{m}I) = d$. Thus $\lambda(I/\mathfrak{m}I) = \lambda(\mathfrak{m}I : x/\mathfrak{m}I)$ and hence $I = \mathfrak{m}I : x$. Therefore I is \mathfrak{m} -full. \square

Theorem 2.4. [9, Theorem 2.4] *Let (R, \mathfrak{m}) be a d -dimensional local ring such that R/\mathfrak{m} is infinite. If $\overline{I} = I$ then I is \mathfrak{m} -full or $I = \sqrt{(0)}$.*

Theorem 2.5. [30, Theorem 1] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified unmixed local ring and I be a parameter ideal. If $\overline{e}_1(I) = e_1(I)$ then R is a regular ring and $\overline{I}^n = I^n$ for all n .*

Proof. Since R is analytically unramified, $\lambda(\overline{\mathfrak{m}^n}/\overline{\mathfrak{m}^{n+1}})$ is a polynomial function of degree $d - 1$ with leading coefficient $e(\mathfrak{m})$. Note that

$$\lambda(\overline{\mathfrak{m}^n}/\overline{\mathfrak{m}^{n+1}}) \leq \lambda(\overline{\mathfrak{m}^n}/\overline{\mathfrak{m}\mathfrak{m}^n}) = \mu(\overline{\mathfrak{m}^n}) \leq \mu(\overline{I}^n).$$

The last inequality holds because \overline{I}^n is \mathfrak{m} -full by Theorem 2.4. We also have

$$\mu(\overline{I}^n) = \lambda(\overline{I}^n/\mathfrak{m}\overline{I}^n) \leq \lambda(\overline{I}^n/\mathfrak{m}I^n) = \lambda(\overline{I}^n/I^n) + \mu(I^n).$$

Since $\overline{e}_1(I) = e_1(I)$, $\lambda(\overline{I}^n/I^n)$ is a polynomial function of degree $< d - 1$ while $\mu(I^n) = \binom{n+d-1}{d-1}$. Thus we have $e(\mathfrak{m}) = 1$. Since R is unmixed by Nagata's Theorem in [31], R is regular. Without loss of generality we may assume that R is a complete local ring.

Let $\overline{\mathcal{R}}(I) = \bigoplus_{n \geq 0} \overline{I}^n t^n$ and $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$. Observe that $\overline{\mathcal{R}}(I)/\mathcal{R}(I)$ is a finite graded $\mathcal{R}(I)$ -module. Since $\overline{e}_1(I) = e_1(I)$ the polynomial function $\lambda(\overline{I}^n/I^n)$ is of degree $< d - 1$. Suppose that there exists n such $\overline{I}^n \neq I^n$. As $\mathcal{R}(I)$ is Cohen-Macaulay, it is the intersection of its localizations at the minimal primes of principal ideals by [23, Theorem 53]. This shows that $\text{ht}[\mathcal{R}(I) :_{\mathcal{R}(I)} \overline{\mathcal{R}}(I)] = 1$. Thus $\dim \overline{\mathcal{R}}(I)/\mathcal{R}(I) = d$. Hence $\lambda(\overline{I}^n/I^n)$ is a polynomial function of degree $d - 1$, which is a contradiction. Therefore $\overline{I}^n = I^n$ for all n . \square

3. THE POSITIVITY CONJECTURE

At a conference held in 2008 in Yokohama, Japan, Wolmer Vasconcelos [40] announced several conjectures about the the Chern number of a parameter ideal and the normal Chern number of an \mathfrak{m} -primary ideal in a Noetherian local ring (R, \mathfrak{m}) . First we quote his conjecture for the normal Chern number and then sketch a solution of the conjecture.

Conjecture 1. [40, Vasconcelos] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified local ring and let I be an \mathfrak{m} -primary ideal. Then $\bar{e}_1(I) \geq 0$.*

We show that the Positivity Conjecture holds for the filtration $\overline{\mathfrak{m}}^n$ where \mathfrak{m} is the maximal homogeneous ideal of the face ring of a pure simplicial complex Δ . Let Δ be a $(d-1)$ -dimensional simplicial complex. Let f_i denote the number of i -dimensional faces of Δ for $i = -1, 0, \dots, d-1$. Here $f_{-1} = 1$. Let Δ have n vertices $\{v_1, v_2, \dots, v_n\}$. Let x_1, x_2, \dots, x_n be indeterminates over a field k . The ideal I_Δ of Δ is the ideal generated by the square free monomials $x_{a_1}x_{a_2}\dots x_{a_m}$ where $1 \leq a_1 < a_2 < \dots < a_m \leq n$ and $\{v_{a_1}, v_{a_2}, \dots, v_{a_m}\} \notin \Delta$. The face ring of Δ over a field k is defined as $k[\Delta] = k[x_1, x_2, \dots, x_n]/I_\Delta$.

Lemma 3.1. *Let R be a Noetherian ring and I be an ideal of R such that the associated graded ring $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ is reduced. Then $\overline{I^n} = I^n$ for all n .*

Proof. Let $\mathcal{R}(I) = \bigoplus_{n \in \mathbb{Z}} I^n t^n$ denote the extended Rees ring of I . Since $G(I) \simeq \mathcal{R}(I)/(u)$, where $u = t^{-1}$, is reduced, $(u) = P_1 \cap P_2 \cap \dots \cap P_r$ for some height one prime ideals P_1, \dots, P_r of $\mathcal{R}(I)$. Therefore (u) is integrally closed in $\mathcal{R}(I)$. As $P_i \mathcal{R}(I)_{P_i} = (u) \mathcal{R}(I)_{P_i}$ for all i , $\mathcal{R}(I)_{P_i}$ is a DVR for all i . Since u is regular, $\text{Ass}(\mathcal{R}(I)/(u^n)) = \{P_1, P_2, \dots, P_r\}$ for all $n \geq 1$. Thus $(u^n) = \bigcap_{i=1}^r P_i^{(n)}$ is integrally closed. Hence $I^n = (u^n) \cap R$ is integrally closed for all n . \square

Theorem 3.2. *Let Δ be a simplicial complex of dimension $d-1$. Let \mathfrak{m} denote the maximal homogeneous ideal of the face ring $k[\Delta]$ over a field k . Then*

- (1) $\overline{\mathfrak{m}}^n = \mathfrak{m}^n$ for all n .
- (2) $\bar{e}_1(\mathfrak{m}) = e_1(\mathfrak{m}) = df_{d-1} - f_{d-2}$.
- (3) If Δ is pure then $\bar{e}_1(\mathfrak{m}) = e_1(\mathfrak{m}) \geq 0$.

Proof. (1) Since $k[\Delta]$ is a standard graded k -algebra, $G(\mathfrak{m}) = k[\Delta]$. Hence $G(\mathfrak{m})$ is reduced and then by Lemma 3.1, $\overline{\mathfrak{m}}^n = \mathfrak{m}^n$ for all $n \geq 0$.
 (2) Since $\lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \dim_k k[\Delta]_n$, The Hilbert Series of the face ring is

$$H(k[\Delta], t) = \sum_{n=0}^{\infty} \dim_k k[\Delta]_n t^n = \frac{h_0 + h_1 t + \dots + h_s t^s}{(1-t)^d}.$$

Put $h(t) = h_0 + h_1t + \cdots + h_st^s$. The face vector $(f_{-1}, f_1, f_0, \dots, f_{d-1})$ and the h -vector are related by the equation

$$\sum_{i=0}^s h_i t^i = \sum_{i=0}^d f_{i-1} t^i (1-t)^{(d-i)}$$

by [4, Lemma 5.1.8]. Then by [4, Proposition 4.1.9] we have

$$e_1(\mathbf{m}) = \bar{e}_1(\mathbf{m}) = h'(1) = df_{d-1} - f_{d-2}.$$

(3) Now we prove that if Δ is a pure simplicial complex then $df_{d-1} \geq f_{d-2}$. Let σ be a facet. For any $v_i \in \sigma = \{v_1, \dots, v_d\}$, $\sigma \setminus \{v_i\}$ is a $(d-2)$ -dimensional face and $\sigma \setminus \{v_i\}$ are distinct for all $i = 1, \dots, d$. Therefore each facet gives rise to d faces of dimension $d-2$. But different facets may produce same faces of dimension $d-2$. Since Δ is pure, each $(d-2)$ -dimensional face is contained in a facet. Hence $df_{d-1} \geq f_{d-2}$. Therefore $\bar{e}_1(\mathbf{m}) \geq 0$ by Theorem 3.2. \square

The above theorem indicates that the maximal homogeneous ideal of the face ring of a non-pure simplicial complex may have negative Chern number. Indeed, consider the simplicial complex Δ_n with its vertex set as $\{v_1, v_2, \dots, v_{n+2}\}$ where $n \geq 2$ and

$$\Delta_n = \{\{v_1, v_2\}, v_3, \dots, v_{n+2}\}.$$

Then $e_1(\mathbf{m}) = df_{d-1} - f_{d-2} = -n$. Hence we need to add the assumption of unmixedness on the ring in Vasconcelos' Positivity Conjecture.

By a **finite cover** S/R , we mean a ring extension $R \subseteq S$ such that S is a finite R -module. Then S is a Noetherian semilocal ring. We say that the finite cover S/R is **birational** if R is reduced and S is contained in the total ring of fractions of R ; that S/R is of finite length if $\lambda(S/R)$ is finite; and that S/R is Cohen-Macaulay if S is Cohen-Macaulay as an R -module.

Theorem 3.3 (Mandal-Singh-Verma, [26]). *Let (R, \mathbf{m}) be an analytically unramified Noetherian local ring of positive dimension. Then $\bar{e}_1(I) \geq 0$ for all \mathbf{m} -primary ideals I of R if R satisfies any one of the following conditions:*

- (1) R has a finite Cohen-Macaulay cover which is of finite length or is birational,
- (2) $\dim R = 1$,
- (3) The integral closure of R is Cohen-Macaulay as an R -module,
- (4) $\dim R = 2$ and all maximal ideals of the integral closure of R have the same height,

(5) R is a complete local integral domain of dimension 2.

In [10] the solution to the Positivity Conjecture has been given for analytically unramified unmixed local rings. In order to prove the Theorem we need a lemma. First we set up the notation for the lemma. Suppose z_1, z_2, \dots, z_d are indeterminates. Let $R' = R[z_1, \dots, z_d]_{\mathfrak{q}}$ where $\mathfrak{q} = \mathfrak{m}R[z_1, \dots, z_d]$ and take quotient by a general element $x = a_1z_1 + \dots + a_dz_d$, where $I = (a_1, \dots, a_d)$ and consider the ring $D' = R'/(x)$. For a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ we define $\Delta f(n) = f(n) - f(n-1)$. The next lemma is crucial for applying induction on the dimension of the ring. It is proved in [20] and [10].

Lemma 3.4. *Let (R, \mathfrak{m}) be a complete normal local domain. Then with the notation above*

- (1) $\overline{I^n}D' = \overline{I^nD'}$ for large n .
- (2) $\overline{P}_{ID'}(n) = \Delta \overline{P}_{IR'}(n)$.

Theorem 3.5 (Goto-Hong-Mandal, [10]). *Let (R, \mathfrak{m}) be an analytically unramified unmixed local ring of dimension $d \geq 2$. Then for every \mathfrak{m} -primary ideal I ,*

$$\overline{e}_1(I) \geq 0.$$

Proof. We sketch the proof. Without loss of generality we may assume R is complete and I is a parameter ideal. We prove the theorem by induction on d . Let $S = \overline{R}$. For each $\mathcal{P} \in \text{Ass } R$ we put $S(\mathcal{P}) = \overline{R/\mathcal{P}}$. Then $S(\mathcal{P})$ is a module-finite extension of R/\mathcal{P} and we get by [18, Corollary 2.1.13]

$$S = \prod_{\mathcal{P} \in \text{Ass } R} S(\mathcal{P}) \quad \text{and} \quad \overline{I^n} = \overline{I^n S} \cap R$$

for all $n \geq 1$. Consider the map $\chi : R/\overline{I^n} \rightarrow S/\overline{I^n S}$. Then $\ker \chi = \overline{I^n S} \cap R = \overline{I^n}$. So χ is injective. Hence

$$\begin{aligned} \lambda_R(R/\overline{I^n}) \leq \lambda_R(S/\overline{I^n S}) &= \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\overline{I^n S(\mathcal{P})}) \\ &= \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\mathfrak{m}_{S(\mathcal{P})}) \lambda_{S(\mathcal{P})}(S(\mathcal{P})/\overline{I^n S(\mathcal{P})}), \end{aligned}$$

where $\mathfrak{m}_{S(\mathcal{P})}$ denotes the maximal ideal of $S(\mathcal{P})$. As $\dim S(\mathcal{P}) = d$ for each $\mathcal{P} \in \text{Ass } R$, we have

$$\begin{aligned} \bar{e}_0(I, R) = e_0(I, R) = e_0(I, S) &= \sum_{\mathcal{P} \in \text{Ass } R} e_0(I, S(\mathcal{P})) \\ &= \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\mathfrak{m}_{S(\mathcal{P})}) e_0(IS(\mathcal{P}), S(\mathcal{P})) \\ &= \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\mathfrak{m}_{S(\mathcal{P})}) \bar{e}_0(IS(\mathcal{P}), S(\mathcal{P})). \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\leq \lambda_R(S/\overline{I^{n+1}S}) - \lambda_R(R/\overline{I^{n+1}}) \\ &= \left[\bar{e}_1(I, R) - \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\mathfrak{m}_{S(\mathcal{P})}) \bar{e}_1(IS(\mathcal{P}), S(\mathcal{P})) \right] \binom{n+d-1}{d-1} \\ &\quad + \text{terms of lower degree in } n. \end{aligned}$$

Hence

$$\bar{e}_1(I, R) \geq \sum_{\mathcal{P} \in \text{Ass } R} \lambda_R(S(\mathcal{P})/\mathfrak{m}_{S(\mathcal{P})}) \bar{e}_1(IS(\mathcal{P}), S(\mathcal{P})).$$

In order to prove $\bar{e}_1(I, R) \geq 0$, it suffices to show that $\bar{e}_1(IS(\mathcal{P}), S(\mathcal{P})) \geq 0$ for each $\mathcal{P} \in \text{Ass } R$. If $d = 2$, as $S(\mathcal{P})$ is a Cohen-Macaulay local ring, $\bar{e}_1(I, R) \geq 0$. Suppose that $d \geq 3$ and that our assertion holds true for $d-1$. Passing to the ring $S(\mathcal{P})$, we may assume that R is a complete local normal domain. Then we consider the general extension ring $T = R[z_1, z_2, \dots, z_d]$ and $R' = R[z_1, \dots, z_d]_{\mathfrak{q}}$ where $\mathfrak{q} = \mathfrak{m}R[z_1, \dots, z_d]$ and take quotient by a general element $x = a_1z_1 + \dots + a_dz_d$, where $I = (a_1, \dots, a_d)$ and consider the ring $D' = R'/(x)$. Then D' is unmixed and analytically unramified. By using Lemma 3.4 we have $\overline{I^n D'} = \overline{I^n D'}$ for large n and

$$\overline{P}_{ID'}(n) = \Delta \overline{P}_{IR'}(n).$$

Then using induction on d we are done. \square

4. STUDY OF $\bar{e}_2(I)$

In this section we survey results on $\bar{e}_2(I)$. As in case of $\bar{e}_1(I)$, it turns out that $\bar{e}_2(I)$ is also non-negative in analytically unramified Cohen-Macaulay local rings. Itoh has given a lower bound for $\bar{e}_2(I)$ and has given a necessary and sufficient condition for the bound to be attained. We give a proof of this theorem. As a consequence we prove Huneke's theorem which gives a necessary and sufficient condition for vanishing of $\bar{e}_2(I)$ in terms of reduction

number $\bar{r}(I)$. We give a necessary and sufficient condition for $\bar{e}_2(I) = 1$. We also discuss a necessary and sufficient condition for $\bar{r}(I) \leq 2$ in terms of $\bar{e}_2(I)$. As a consequence we derive a theorem of Corso, Polini and Rossi [6, Theorem 3.12] which gives a condition for a normal ideal to have reduction number two.

T. Marley in his thesis [27, Proposition 3.23] proved that $e_2(\mathcal{I}) \geq 0$ for any I -admissible filtration \mathcal{I} in a Cohen-Macaulay local ring.

Theorem 4.1. [27, Proposition 3.23] *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Then $e_2(\mathcal{I}) \geq 0$.*

We give a proof of positivity of $\bar{e}_2(I)$. In order to prove this we will use the following results.

Theorem 4.2. [2, Theorem 4.1] *Let (R, \mathfrak{m}) be a d -dimensional local ring and I be an \mathfrak{m} -primary ideal. Let $\mathcal{I} = \{I_n\}$ be an I -admissible filtration. Let $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$. Then*

- (1) *For all $i \geq 0$, $\lambda([H_{\mathcal{R}_+}^i(\mathcal{R}(\mathcal{I}))]_n) < \infty$ for all $n \in \mathbb{Z}$.*
- (2) *For all $n \in \mathbb{Z}$,*

$$P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n) = \sum_{i=0}^d (-1)^i \lambda([H_{\mathcal{R}_+}^i(\mathcal{R}(\mathcal{I}))]_n).$$

The following result of Itoh is a key point in proving results about normal Hilbert polynomials using induction on the dimension of R . See also [5].

Theorem 4.3. [20, Theorem 1 and Corollary 8] *Let I a parameter ideal of a Cohen-Macaulay analytically unramified local ring (R, \mathfrak{m}) of dimension d . Then there exists a system of generators x_1, \dots, x_d of I such that, if we put $C = R(T)/(\sum_i x_i T_i)$ and $J = IC$, where $R(T) = R[T]_{\mathfrak{m}[T]}$ and $T = (T_1, \dots, T_d)$ is a set of d indeterminates. Then*

- (1) $\overline{J^n} \cap R = \overline{I^n}$ for every $n \geq 0$,
- (2) $\overline{J} = \overline{I}C$,
- (3) $\overline{J^n} = \overline{I^n}C \cong \overline{I^n}R(T)/(\sum_i x_i T_i) \overline{I^{n-1}}R(T)$ for large n ,
- (4) *If R is analytically normal and $\dim R \geq 3$, then C is normal.*
- (5) $\bar{e}_i(I) = \bar{e}_i(J)$ for $i = 0, 1, \dots, d-1$.

We quote some results of Itoh about vanishing of graded components of local cohomology modules. See also a recent paper of Hong and Ulrich [11].

Theorem 4.4. [20, Proposition 13] *Let (R, \mathfrak{m}) be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$. Let $N = It\mathcal{R}(I)$ and $M = (t^{-1}, It)\mathcal{R}(I)$. Then*

- (1) $H_M^0(\overline{\mathcal{R}}(I)) = H_M^1(\overline{\mathcal{R}}(I)) = 0$,
- (2) $[H_M^2(\overline{\mathcal{R}}(I))]_j = 0$ for $j \leq 0$,
- (3) $H_M^i(\overline{\mathcal{R}}(I)) = H_N^i(\overline{\mathcal{R}}(I))$ for $i = 0, 1, \dots, d-1$.

Now we prove nonnegativity of $\bar{e}_2(I)$. For a generalisation to good filtrations in Cohen-Macaulay modules, see [36, Proposition 3.1].

Theorem 4.5. *Let (R, \mathfrak{m}) be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ and let I be an \mathfrak{m} -primary ideal. Then*

$$\bar{e}_2(I) \geq 0.$$

Proof. We apply induction on d . Let $d = 2$. By Theorem 4.2,

$$\overline{P}_I(n) - \overline{H}_I(n) = \sum_{i=0}^2 (-1)^i \lambda([H_N^i(\overline{\mathcal{R}}(I))]_n)$$

for all $n \in \mathbb{Z}$. Taking $n = 0$ we get $\bar{e}_2(I) = \lambda([H_N^2(\overline{\mathcal{R}}(I))]_0) \geq 0$. Let $d > 2$. Let C and J be as in Theorem 4.3. Then by induction hypothesis $\bar{e}_2(I) = \bar{e}_2(J) \geq 0$. \square

The following theorem of Itoh gives a lower bound on $\bar{e}_2(I)$. For a generalisation for good filtration of modules see [36, Theorem 3.1].

Theorem 4.6. [20, Theorem 2(2)] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified Cohen-Macaulay local ring. Let I be a parameter ideal. Then*

$$\bar{e}_2(I) \geq \bar{e}_1(I) - \lambda(\overline{I}/I).$$

Proof. Apply induction on d . For $d = 2$, by Theorems 4.2 and 4.4 ,

$$\overline{P}_I(1) - \overline{H}_I(1) = \lambda([H_N^2(\overline{\mathcal{R}}(I))]_1) \geq 0.$$

Hence $\bar{e}_0(I) - \bar{e}_1(I) + \bar{e}_2(I) \geq \lambda(R/\overline{I})$. Since I is a parameter ideal, $\bar{e}_0(I) = \lambda(R/I)$. Therefore $\bar{e}_2(I) \geq \bar{e}_1(I) - \lambda(\overline{I}/I)$. Let $d > 2$. Let C and J be as in Theorem 4.3. Since $\overline{J} = \overline{I}C$, $\lambda(\overline{I}/I) = \lambda(\overline{J}/J)$. Hence the inequality follows by induction hypothesis. \square

Theorem 4.7. *Under the assumptions of Theorem 4.6,*

- (1) [20, Theorem 2(2)] $\bar{e}_2(I) = \bar{e}_1(I) - \lambda(\overline{I}/I)$ if and only if $\overline{r}(I) \leq 2$.

(2) [6, Theorem 3.12] If $\bar{r}(I) \leq 2$ then $\bar{G}(I)$ is Cohen-Macaulay and its Hilbert series is given by

$$\bar{F}_I(t) = \frac{\lambda(R/\bar{I}) + [\bar{e}_0(I) - \lambda(R/\bar{I}) - \lambda(\bar{I}^2/I\bar{I})]t + \lambda(\bar{I}^2/I\bar{I})t^2}{(1-t)^d}.$$

Proof. Let $\bar{r}(I) \leq 2$. By Huckaba-Marley Theorem,

$$\sum_{n \geq 1} \lambda(\bar{I}^n/I \cap \bar{I}^n) \leq \bar{e}_1(I) \leq \sum_{n \geq 1} \lambda(\bar{I}^n/I\bar{I}^{n-1}) = \lambda(\bar{I}/I) + \lambda(\bar{I}^2/I\bar{I}).$$

Using Huneke-Itoh Intersection theorem we get $I \cap \bar{I}^2 = I\bar{I}$. Therefore

$$\bar{e}_1(I) = \sum_{n \geq 1} \lambda(\bar{I}^n/I \cap \bar{I}^n) = \lambda(\bar{I}/I) + \lambda(\bar{I}^2/I\bar{I}).$$

Hence by Theorem 1.2, $\bar{G}(I)$ is Cohen-Macaulay. Let $I = (x_1, x_2, \dots, x_d)$. Let x_i^* denote the image of x_i in \bar{I}/\bar{I}^2 . Let $\mathcal{F} = \{\bar{I}^n + I/I\}$. Then

$$\begin{aligned} \bar{G}(I)/(x_1^*, \dots, x_d^*) &= G(\mathcal{F}) \\ &= \bigoplus_{n \geq 0} \frac{\bar{I}^n + I}{\bar{I}^{n+1} + I} \\ &= \frac{R}{\bar{I}} \oplus \frac{\bar{I}}{\bar{I}^2 + I} \oplus \frac{\bar{I}^2 + I}{\bar{I}^3 + I}. \end{aligned}$$

Therefore

$$H(\bar{G}(I), t) = \frac{\lambda(R/\bar{I}) + \lambda(\bar{I}/\bar{I}^2 + I)t + \lambda(\bar{I}^2 + I/I)t^2}{(1-t)^d}.$$

Hence

$$\bar{e}_2(I) = \lambda(\bar{I}^2 + I/I) = \lambda(\bar{I}^2/I \cap \bar{I}^2) = \lambda(\bar{I}^2/I\bar{I}).$$

Also $\bar{e}_1(I) = \lambda(\bar{I}/I) + \lambda(\bar{I}^2/I\bar{I})$. Therefore $\bar{e}_2(I) = \bar{e}_1(I) - \lambda(\bar{I}/I)$.

Conversely let $\bar{e}_2(I) = \bar{e}_1(I) - \lambda(\bar{I}/I)$. Use induction on d . Let $d = 2$. Since $\lambda(R/\bar{I}) = \bar{e}_0(I) - \bar{e}_1(I) + \bar{e}_2(I)$, $\bar{P}_I(1) = \bar{H}_I(1)$. Therefore by Theorems 4.2 and 4.4, $\lambda([H_N^2(\bar{\mathcal{R}}(I))]_1) = 0$. By [3, Lemma 4.3.5], $\lambda([H_N^2(\bar{\mathcal{R}}(I))]_n) = 0$ for all $n \geq 1$. Hence $\bar{P}_I(n) = \bar{H}_I(n)$ for all $n \geq 1$. Hence $\bar{n}(I) \leq 0$. By Theorem 1.7, $\bar{r}(I) \leq 2$. Consider C as and J as in theorem 4.3. Then by induction $\bar{J}^{n+2} = J^n \bar{J}^2$ for all $n \geq 0$. Therefore by [20, Proposition 17] $\bar{I}^{n+2} = I^n \bar{I}^2$ for all $n \geq 0$. Hence $\bar{r}(I) \leq 2$. \square

Now we analyse vanishing of $\bar{e}_2(I)$. By Theorem 1.5, $\lambda(R/I_1) = e_0(\mathcal{F}) - e_1(\mathcal{F})$ if and only if $r(\mathcal{F}) \leq 1$. In this case, $G(\mathcal{F})$ is Cohen-Macaulay by Huckaba-Marley Theorem. Hence $e_2(\mathcal{F}) = 0$. For integral closure filtration, $\mathcal{F} = \{\bar{I}^n\}$, converse is also true. In other words vanishing of $\bar{e}_2(I)$ is sufficient to guarantee that $\bar{r}(I) \leq 1$. Huneke [15, Theorem 4.5] proved this if $d = 2$.

Theorem 4.8. *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified Cohen-Macaulay local ring with infinite residue field and let I be an \mathfrak{m} -primary ideal. Then $\bar{e}_2(I) = 0$ if and only if $\bar{r}(I) \leq 1$.*

Proof. Let $\bar{e}_2(I) = 0$. Let J be a minimal reduction of I . By Theorem 4.6 and Theorem 1.10,

$$0 = \bar{e}_2(J) \geq \bar{e}_1(J) - \lambda(\bar{J}/J) \geq \lambda(\bar{J}^2/J\bar{J}).$$

Therefore $\lambda(\bar{J}^2/J\bar{J}) = 0 = \bar{e}_1(J) - \lambda(\bar{J}/J)$. Hence $\bar{e}_1(I) = \lambda(\bar{I}/I)$. By Theorems 1.5, $\bar{r}(I) \leq 1$. Conversely let $\bar{r}(I) \leq 1$. We may assume that I is a parameter ideal. Then by Theorem 4.7, $\bar{e}_2(I) = \bar{e}_1(I) - \lambda(\bar{I}/I)$. By Theorem 1.5, $\bar{e}_1(I) - \lambda(\bar{I}/I) = 0$. So $\bar{e}_2(I) = 0$. \square

Next proposition gives a necessary and sufficient condition for $\bar{e}_2(I) = 1$.

Proposition 4.9. [21, Theorem 9] *Let (R, \mathfrak{m}) be an analytically unramified Cohen Macaulay local ring of dimension d . Let I be a parameter ideal. Then $\bar{e}_1(I) = \bar{e}_0(I) - \lambda(R/\bar{I}) + 1$ if and only if $\bar{e}_2(I) = 1$. In this case, $\bar{r}(I) = 2$ and*

$$\bar{F}_I(t) = \frac{\lambda(R/\bar{I}) + [\bar{e}_0(I) - \lambda(R/\bar{I}) + 1]t + t^2}{(1-t)^d}.$$

Proof. Let $\bar{e}_1(I) = \bar{e}_0(I) - \lambda(R/\bar{I}) + 1$. We use induction on d to prove that $\bar{e}_2(I) = 1$. Let $d = 2$. Let $J \subseteq I$ be a minimal reduction of I . Then by Huckaba-Marley Theorem,

$$\sum_{n \geq 2} \lambda(\bar{J}^n/J \cap \bar{J}^n) \leq \bar{e}_1(J) - \lambda(\bar{J}/J).$$

Therefore $\sum_{n \geq 2} \lambda(\bar{J}^n/J \cap \bar{J}^n) \leq 1$. Suppose $\sum_{n \geq 2} \lambda(\bar{J}^n/J \cap \bar{J}^n) = 0$. Then $\bar{J}^2 = J \cap \bar{J}^2 = J\bar{J}$, by Huneke-Itoh intersection theorem. By [27, Theorem 3.26] $\bar{e}_2(J) = 0$. But Theorem 4.6 implies that

$$\bar{e}_2(J) \geq \bar{e}_1(J) - \lambda(\bar{J}/J) = \bar{e}_1(I) - \bar{e}_0(I) + \lambda(R/\bar{I}) = 1,$$

a contradiction. Hence $\sum_{n \geq 2} \lambda(\bar{J}^n/J \cap \bar{J}^n) = 1$. Therefore by Theorem 1.2, $\bar{G}(I)$ is Cohen-Macaulay. Similar argument as above shows that $\bar{J}^2 \neq J\bar{J}$. Therefore $\bar{J}^n = J \cap \bar{J}^n$ for all $n \geq 3$. Hence by Valabrega-Valla theorem, $\bar{J}^n = J\bar{J}^{n-1}$ for all $n \geq 3$. Therefore Theorem 4.7 implies that $\bar{e}_2(J) = \bar{e}_1(J) - \lambda(\bar{J}/J) = 1$. Hence $\bar{e}_2(I) = 1$. Let $d > 2$. We may assume that I is a parameter ideal. Let C and J be as in theorem 4.3. Hence

$\bar{e}_1(J) = \bar{e}_0(J) - \lambda(C/\bar{J}) + 1$. Therefore by induction, $\bar{e}_2(J) = 1$. Hence $\bar{e}_2(I) = 1$.

Conversely, let $\bar{e}_2(I) = 1$. Let J be a minimal reduction of I . Then by Theorem 4.6 and 1.10,

$$1 = \bar{e}_2(J) \geq \bar{e}_1(J) - \lambda(\bar{J}/J) \geq 0.$$

Suppose $\bar{e}_1(J) - \lambda(\bar{J}/J) = 0$. Then Huckaba-Marley Theorem implies that $\bar{e}_1(J) = \sum_{n \geq 1} \lambda(\bar{J}^n/J \cap \bar{J}^n)$ and $\bar{G}(I)$ is Cohen-Macaulay. Therefore $\bar{J}^n = J \cap \bar{J}^n$ for all $n \geq 2$. By Valabrega-Valla theorem, $\bar{J}^n = J\bar{J}^{n-1}$ for all $n \geq 2$. Hence $\bar{r}(J) \leq 1$. By Theorem 4.8, $\bar{e}_2(J) = 0$, a contradiction. Therefore $\bar{e}_1(J) - \lambda(\bar{J}/J) = 1$ and hence $\bar{e}_1(I) = \bar{e}_0(I) - \lambda(R/\bar{I}) + 1$. \square

As a consequence of Theorem 4.7 we get similar result for normal ideals in [6, Theorem 3.12]. However the following example in [6] shows that $\lambda(R/I) = e_0(I) - e_1(I) + e_2(I)$ is not a sufficient condition to guarantee that $G(I)$ is Cohen-Macaulay.

Example 4.10. [6, Example 3.10] Let (R, \mathfrak{m}) be the 2-dimensional local Cohen-Macaulay ring

$$k[[x, y, z, u, v]]/(z^2, zu, zv, uv, yz - u^3, xz - v^3)$$

with k a field and x, y, z, u, v indeterminate. One can see that the depth $G(\mathfrak{m}) = 0$ and

$$F_{\mathfrak{m}}(t) = \frac{1 + 3t + 3t^3 - t^4}{(1 - t)^2}.$$

In this case $e_2(\mathfrak{m}) = e_1(\mathfrak{m}) - e_0(\mathfrak{m}) + 1$ but $G(\mathfrak{m})$ is not Cohen-Macaulay.

5. STUDY OF $\bar{e}_3(I)$

So far we have seen that $e_1(\mathcal{I})$ and $e_2(\mathcal{I})$ are non-negative for any I -admissible filtration in a Cohen-Macaulay local ring. But the non-negativity does not hold true for $e_3(\mathcal{I})$. Note that $e_3(\mathfrak{m}) = -1$ in Example 4.10. Marley in [27] has given an another example in which $e_3(I)$ is negative.

Example 5.1. Let $R = k[[x, y, z]]$ where k is a field. Then the Hilbert polynomial $P_I(x)$ of the ideal

$$I = (x^3, y^3, z^3, x^2y, xy^2, yz^2, xyz)$$

is given by

$$P_I(x) = 27 \binom{x+2}{3} - 18 \binom{x+1}{2} + 4x + 1.$$

Hence $e_3(I) = -1$.

However Itoh has proved that $\bar{e}_3(I)$ is non-negative in a Cohen-Macaulay analytically unramified local ring. He has proposed a conjecture for the vanishing of $\bar{e}_3(I)$ in [20].

Itoh's Conjecture: Let (R, \mathfrak{m}) be a analytically unramified Gorenstein local ring of dimension $d \geq 3$. Then $\bar{e}_3(I) = 0$ if and only if $\bar{r}(I) \leq 2$.

Itoh has given a solution to the conjecture when $\bar{I} = \mathfrak{m}$. In order to prove this first we recall some preliminary results.

Proposition 5.2. [19, Proposition 10] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified Cohen-Macaulay local ring and let I be a parameter ideal. Then for all $n \geq 0$,*

$$\begin{aligned} \lambda(R/\bar{I}^{n+1}) &\leq \lambda(R/I) \binom{n+d}{d} - [\lambda(\bar{I}/I) + \lambda(\bar{I}^2/I\bar{I})] \binom{n+d-1}{d-1} \\ &\quad + \lambda(\bar{I}^2/I\bar{I}) \binom{n+d-2}{d-2}. \end{aligned}$$

Moreover, equality holds for all $n \geq 1$ if and only if $\bar{r}(I) \leq 2$. In particular if $\bar{r}(I) \leq 2$ then $\bar{e}_i(I) = 0$ for $i \geq 3$.

Proof. If $n = 0$ then the inequality trivially holds. So assume $n \geq 1$. Since $I^n \bar{I} \subseteq I^{n-1} \bar{I}^2 \subseteq \bar{I}^{n+1}$ we have $\lambda(R/I^{n+1}) \leq \lambda(R/I^n \bar{I}) - \lambda(I^{n-1} \bar{I}^2/I^n \bar{I})$. Since $I^n/I^{n+1} \otimes R/\bar{I} \cong I^n/I^n \bar{I}$ and I^n/I^{n+1} is a free R/I module we get

$$\begin{aligned} \lambda(R/I^n \bar{I}) &= \lambda(R/I^n) + \lambda(I^n/I^n \bar{I}) \\ &= \lambda(R/I) \binom{n-1+d}{d} + \lambda(R/\bar{I}) \binom{n+d-1}{d-1} \\ &= \lambda(R/I) \binom{n+d}{d} - \lambda(\bar{I}/I) \binom{n+d-1}{d-1}. \end{aligned}$$

Therefore it is sufficient to prove that $\lambda(I^{n-1} \bar{I}^2/I^n \bar{I}) = \lambda(\bar{I}^2/I\bar{I}) \binom{n-1+d-1}{d-1}$. Since $I^n \bar{I} \subseteq I^{n-1} \bar{I}^2 \cap I^n \subseteq \bar{I}^{n+1} \cap I^n = I^n \bar{I}$ (by Theorem 1.8), we have $I^n \bar{I} = I^{n-1} \bar{I}^2 \cap I^n$ and hence $I^{n-1} \bar{I}^2/I^n \bar{I} \cong (I^{n-1} \bar{I}^2 + I^n)/I^n$.

Now we prove that the canonical morphism

$$I^{n-1}/I^n \otimes \bar{I}^2/I\bar{I} \longrightarrow (I^{n-1} \bar{I}^2 + I^n)/I^n$$

is an isomorphism. Let x_1, \dots, x_d be a regular sequence such that $I = (x_1, \dots, x_d)$ and let $\{M_j\}$ be the set of monomials in x_1, \dots, x_d of degree $n-1$. It is sufficient to show that if $\sum_j a_j M_j \in I^n$ with $a_j \in \bar{I}^2$ then $a_j \in I$ and hence $a_j \in \bar{I}^2 \cap I = I\bar{I}$. Suppose that $\sum_j a_j M_j = \sum_j b_j M_j$ with $b_j \in I$.

Since x_1, \dots, x_d is a regular sequence we have $a_j - b_j \in I$ for each j and hence $a_j \in I$. Therefore $(I^{n-1}\overline{I^2} + I^n)/I^n \cong I^{n-1}/I^n \otimes \overline{I^2}/I\overline{I}$. \square

Let $\underline{x} = x_1, \dots, x_d$ be a minimal generators of I . Then the natural exact sequence of Čech-complexes,

$$0 \longrightarrow C(\underline{x}; R)[t, t^{-1}] \longrightarrow C(t^{-1}, \underline{x}; \overline{\mathcal{R}}(I)) \longrightarrow C(\underline{x}; \overline{\mathcal{R}}(I)) \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow H_{\mathcal{M}}^d(\overline{\mathcal{R}}(I)) \longrightarrow H_{\mathcal{R}_+}^d(\overline{\mathcal{R}}(I)) \longrightarrow H_{\mathfrak{m}}^d(R)[t, t^{-1}] \longrightarrow H_{\mathcal{M}}^{d+1}(\overline{\mathcal{R}}(I)) \longrightarrow 0.$$

Consider the the canonical graded homomorphism

$$\alpha : H_{\mathcal{R}_+}^d(\overline{\mathcal{R}}(I)) \longrightarrow H_{\mathfrak{m}}^d(R)[t, t^{-1}].$$

We denote by α_j the graded part of degree j of α . Then we have

Lemma 5.3. [20, Lemma 18] *The map $\alpha_j = 0$ if and only if for all $n \geq 0$,*

$$\overline{I^{n+d-1+j}} \subseteq I^n.$$

Proof. Let x_1, \dots, x_d be a system of minimal generators of I . We have a natural morphism of Čech complexes and its cohomologies:

$$\begin{array}{ccccccc} [\prod_i \overline{\mathcal{R}}(I)_{x_1 t \dots (x_i t) \wedge \dots x_d t}]_j & \longrightarrow & [\overline{\mathcal{R}}(I)_{x_1 t \dots x_d t}]_j & \longrightarrow & [H_{\mathcal{R}_+}^d(\overline{\mathcal{R}}(I))]_j & \longrightarrow & 0 \\ \downarrow & & \downarrow \beta_j & & \downarrow \alpha_j & & \\ \prod_i R_{x_1 \dots (x_i) \wedge \dots x_d} & \xrightarrow{f} & R_{x_1 \dots x_d} & \longrightarrow & H_{\mathfrak{m}}^d(R) & \longrightarrow & 0 \end{array}$$

Let $\overline{I^{n+d-1+j}} \subseteq I^n$ for all $n \geq 0$. Let

$$z = at^j t^{nd} / (x_1 t \dots x_d t)^n \in [\overline{\mathcal{R}}(I)_{x_1 t \dots x_d t}]_j.$$

Therefore $a \in \overline{I^{nd+j}} \subseteq I^{nd+j-(d-1+j)} = I^{nd-d+1}$. Let $x^c = x_1^{c_1} \dots x_d^{c_d}$ for $(c_1, \dots, c_d) \in \mathbb{N}^d$ and

$$a = \sum_{c_1 + \dots + c_d = nd-d-1} r_c x^c.$$

If $c_i < n$ for all i , then $c_1 + \dots + c_d \leq d(n-1) < nd-d+1$. Therefore $c_i \geq n$ for some i . Hence $a \in (x_1^n, \dots, x_d^n)$. Let $a = \sum_{i=1}^d r_i x_i^n$. Then

$$f \left(\sum_{i=1}^d (-1)^{i-1} r_i / (x_1 \dots \hat{x}_i \dots x_d)^n \right) = a / (x_1 \dots x_d)^n.$$

Thus $\beta_j(z) = a/(x_1 \cdots x_d)^n \in \text{image } f$ and hence $\alpha_j \equiv 0$. Conversely, let $\alpha_j \equiv 0$. Since

$$I^n = \bigcap_{c_1 + \cdots + c_d = n+d-1} (x_1^{c_1}, \dots, x_d^{c_d}),$$

it is sufficient to show that $\overline{I^{n+d-1+j}}$ is contained in $(x_1^{c_1}, \dots, x_d^{c_d})$ for all integers $c_i \geq 1$ with $c_1 + \cdots + c_d = n + d - 1$. Let $a \in \overline{I^{n+d-1+j}}$. Then $z = at^j/(x_1^{c_1} \cdots x_d^{c_d}) \in [\overline{\mathcal{R}(I)}_{x_1 t \cdots x_d t}]_j$. Therefore $\beta_j(z) \in \text{image } f$. Let

$$\beta_j(z) = a/(x_1^{c_1} \cdots x_d^{c_d}) = \sum (a_i x_i^s)/(x_1 \cdots x_d)^s$$

for some $a_i \in R$ and $s \geq \max\{c_i \mid i = 1, \dots, d\}$. Since $ax_1^{s-c_1} \cdots x_d^{s-c_d} = \sum a_i x_i^s$ we get $a \in (x_1^{c_1}, \dots, x_d^{c_d})$. This proves the assertion. \square

Proposition 5.4. [20, Proposition 19] *Let (R, \mathfrak{m}) be a Gorenstein, analytically unramified local ring of positive dimension. Let I be a parameter ideal such that $\overline{I^2} \neq I\overline{I}$. If $\overline{I^{n+2}}$ and $\mathfrak{m}\overline{I^{n+1}}$ are contained in I^n for every $n \geq 0$, then $\lambda(\overline{I^2}/I\overline{I}) = 1$ and $\overline{r}(I) \leq 2$.*

Proof. Let $I = (x_1, \dots, x_d)$. Since $I^n \supseteq \mathfrak{m}\overline{I^{n+1}}$, using Huneke-Itoh Intersection Theorem

$$(I^n : \mathfrak{m})/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n \cap \overline{I^{n+1}} = \overline{I^{n+1}}/I^n \overline{I}.$$

Therefore $\lambda(\overline{I^{n+1}}/I^n \overline{I}) \leq \lambda((I^n : \mathfrak{m})/I^n)$. As R is Gorenstein, $(x_1^{c_1}, \dots, x_d^{c_d})$ is an irreducible ideal. Hence

$$I^n = \bigcap_{c_1 + \cdots + c_d = n+d-1} (x_1^{c_1}, \dots, x_d^{c_d})$$

is an irredundant decomposition of I^n as a product of irreducible ideals where $c_1, c_2, \dots, c_d \geq 1$. The dimension of $(I^n : \mathfrak{m})/I^n$ as R/\mathfrak{m} -vector space equals the number of irreducible components of I^n . Hence $\lambda((I^n : \mathfrak{m})/I^n) = \binom{n-1+d-1}{d-1}$. Therefore $\lambda(\overline{I^{n+1}}/I^n \overline{I}) \leq \binom{n-1+d-1}{d-1}$. Now

$$\begin{aligned} \lambda(R/\overline{I^{n+1}}) &= \lambda(R/I^{n+1}) - \lambda(\overline{I^{n+1}}/I^{n+1}) \\ &= \lambda(R/I^{n+1}) - \lambda(I^n \overline{I}/I^{n+1}) - \lambda(\overline{I^{n+1}}/I^n \overline{I}). \end{aligned}$$

But

$$\begin{aligned} \lambda(I^n \overline{I}/I^{n+1}) &= \lambda(R/I^{n+1}) - \lambda(R/I^n \overline{I}) \\ &= \lambda(R/I^{n+1}) - \lambda(R/I^n) - \lambda(I^n/I^n \overline{I}) \\ &= \lambda(I^n/I^{n+1}) - \lambda(I^n/I^n \overline{I}). \end{aligned}$$

Since $I^n/I^{n+1} \otimes R/\bar{I} \cong I^n/I^n\bar{I}$, for all $n \geq 0$,

$$\lambda(I^n/I^n\bar{I}) = \lambda(R/\bar{I}) \binom{n+d-1}{d-1}.$$

Hence $\lambda(I^n\bar{I}/I^{n+1}) = \lambda(\bar{I}/I) \binom{n+d-1}{d-1}$. Therefore

$$\begin{aligned} \lambda(R/\overline{I^{n+1}}) &\geq \lambda(R/I) \binom{n+d}{d} - \lambda(\bar{I}/I) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1} \\ &= \lambda(R/I) \binom{n+d}{d} - [\lambda(\bar{I}/I) + 1] \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2} \end{aligned}$$

By Proposition 5.2, for all $n \geq 0$,

$$\begin{aligned} \lambda(R/\overline{I^{n+1}}) &\leq \lambda(R/I) \binom{n+d}{d} - [\lambda(\bar{I}/I) + \lambda(\bar{I}^2/I\bar{I})] \binom{n+d-1}{d-1} \\ &\quad + \lambda(\bar{I}^2/I\bar{I}) \binom{n+d-2}{d-2}. \end{aligned}$$

Hence $-\binom{n+d-2}{d-1} \leq -\lambda(\bar{I}^2/I\bar{I}) \binom{n+d-2}{d-1}$. Therefore $\lambda(\bar{I}^2/I\bar{I}) = 1$ and equality holds in Proposition 5.2. Hence $\bar{r}(I) \leq 2$. \square

Theorem 5.5. [20, Theorem 3] *Let (R, \mathfrak{m}) be a d -dimensional analytically unramified Cohen-Macaulay local ring and let I be a parameter ideal. Suppose $d \geq 3$. Then*

- (1) $\bar{e}_3(I) \geq 0$ and if $\bar{e}_3(I) = 0$ then $\overline{I^{n+2}} \subseteq I^n$ for every $n \geq 0$.
- (2) If R is Gorenstein and $\bar{I} = \mathfrak{m}$ then $\bar{e}_3(I) = 0$ if and only if $\bar{r}(I) \leq 2$.

Proof. (1) Apply induction on d . Assume $d = 3$. By Theorem 4.2 and Theorem 4.4, we have $\bar{e}_3(I) = \lambda(H_{\mathcal{R}_+}^3(\overline{\mathcal{R}(I)})_0) \geq 0$. If $\bar{e}_3(I) = 0$ then $[H_{\mathcal{R}_+}^3(\overline{\mathcal{R}(I)})]_0 = 0$. Therefore $\alpha_0 = 0$ in lemma 5.3. Hence $\overline{I^{n+2}} \subseteq I^n$ for every $n \geq 0$. Using Theorem 4.3 it is easy to see that the result holds in higher dimension also.

(2) Let R be Gorenstein and $\bar{I} = \mathfrak{m}$. Let $\bar{e}_3(I) = 0$. Then $\mathfrak{m}\overline{I^{n+1}} \subseteq \overline{I^{n+2}} \subseteq I^n$ for all $n \geq 0$. By proposition 5.4, $\overline{I^{n+2}} = I^n\bar{I}^2$ for every $n \geq 0$. The converse follows from Proposition 5.2. \square

Huckaba-Huneke in [14] gave a different proof of the non-negativity of $\bar{e}_3(I)$ by reducing to dimension three and then applying the following theorem.

Theorem 5.6. [14, Theorem 3.1] *Let (R, \mathfrak{m}) a d -dimensional Cohen-Macaulay local ring. Let I be a normal ideal with $\text{grade}(I) \geq 2$ and I be integral over*

an ideal generated by an R -regular sequence. Then there exists n such that $\text{depth } G(I^n) \geq 2$.

In [6] Corso, Polini and Rossi showed that in an analytically unramified Cohen-Macaulay local ring of dimension three if $\bar{e}_3(I) = 0$ for an \mathfrak{m} -primary ideal then $G(I^n)$ is Cohen-Macaulay for large n .

Theorem 5.7. [6, Corollary 4.5] *Let (R, \mathfrak{m}) be a local Cohen-Macaulay ring of dimension three with infinite residue field. Let I be an \mathfrak{m} -primary ideal of R such that I is asymptotically normal. Then $e_3(I) = 0$ if and only if there exists some n such that the reduction number of I^n is at most two. Under these conditions, $G(I^n)$ is Cohen-Macaulay for $n \gg 0$.*

The following example from [6] shows that if an ideal I is asymptotically normal such that $e_3(I) = 0$ then $G(I)$ need not be Cohen-Macaulay.

Example 5.8. Let $R = k[[x, y, z]]$ with k a field and x, y, z indeterminates. Consider the ideal $I = (x^2 - y^2, y^2 - z^2, xy, xz, yz)$. Then I^n is integrally closed for every $n \geq 2$. We also have

$$F_I(t) = \frac{5 + 6t^2 - 4t^3 + t^4}{(1 - t)^3}$$

which gives $e_3(I) = 0$ but $G(I)$ has depth zero. On the other hand I^2 is a normal ideal with $e_3(I^2) = 0$ and $G(I^2)$ is Cohen-Macaulay and reduction number is two.

Huckaba and Huneke in [14] have shown that in a two dimensional analytically unramified Cohen-Macaulay local ring, for some n , $\overline{G}(I^n)$ is Cohen-Macaulay.

Theorem 5.9. [14, Theorem 3.1] *Let (R, \mathfrak{m}) be a two dimensional Cohen-Macaulay local ring with infinite residue field and let I be an normal \mathfrak{m} -primary ideal of R . Then there exists n such that $G(I^n)$ is Cohen-Macaulay.*

They have also shown that the above result cannot be extended to higher dimension in view of

Example 5.10. Let k be a field of characteristic $\neq 3$. Set $R = k[[x, y, z]]$. Let

$$N = (x^4, x(y^3 + z^3), y(y^3 + z^3), z(y^3 + z^3))$$

and set $I = N + \mathfrak{m}^5$ where $\mathfrak{m} = (x, y, z)$. Then I is a height 3 normal ideal of R and $G(I^n)$ is not Cohen-Macaulay for any $n \geq 1$.

6. NORMAL HILBERT POLYNOMIALS IN TWO DIMENSIONAL REGULAR LOCAL RINGS

Throughout this section, we assume that (R, \mathfrak{m}) is a regular local ring of dimension two unless otherwise stated.

In this section we present a formula for the normal Hilbert polynomial of two \mathfrak{m} -primary ideals of R . This is a consequence of a result of Lipman and Teissier [25] that for any complete \mathfrak{m} -primary ideal in R , $\bar{r}(I) \leq 1$. The formula has also been derived in [24], [16] and [35]. We shall use joint reductions and Zariski's theory of complete ideals in R to derive this formula. We shall also derive the same formula using a formula of Hoskin and Deligne for the colength of a complete \mathfrak{m} -primary ideal.

A crucial step for obtaining a formula for the normal Hilbert polynomial of two ideals is to show that if I and J are complete \mathfrak{m} -primary ideals in R with R/\mathfrak{m} infinite then there exist $a \in I$ and $b \in J$ such that $aI + bJ = IJ$. This was proved in [41]. By taking $I = J$ we get Lipman-Teissier formula: $I^2 = (a, b)I$.

Joint reductions were introduced by Rees in [34]. Let I_1, I_2, \dots, I_r be ideals in a local ring (R, \mathfrak{m}) . A set of elements (x_1, \dots, x_r) such that $x_i \in I_i$ is called a joint reduction of the set of ideals (I_1, \dots, I_r) if there are positive integers a_1, \dots, a_r such that

$$\sum_{i=1}^r x_i I_1^{a_1} I_2^{a_2} \dots I_i^{a_i-1} \dots I_r^{a_r} = I_1^{a_1} \dots I_r^{a_r}.$$

Rees showed [34] that if R/\mathfrak{m} is infinite and I_1, \dots, I_r are \mathfrak{m} -primary ideals in any local ring (R, \mathfrak{m}) then joint reductions exist. Liam O'Carroll showed the existence of joint reductions for the arbitrary ideals [32].

We now recall a few facts from Zariski's theory of complete ideals. An ideal I of R is called **contracted** if there is an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $IR[\mathfrak{m}/x] \cap R = I$. Here $R[\mathfrak{m}/x] = R[y/x]$ where $\mathfrak{m} = (x, y)$. By [44, Lemma 3, Appendix 5], if I and J are contracted from $R[\mathfrak{m}/x]$ then so is IJ . Rees [35, Lemma 3.1] and Lipman [24, Corollary 3.2] showed that if I is \mathfrak{m} -primary and contracted then $\mu(I) = 1 + o(I)$ where $\mu(I) = \dim(I/\mathfrak{m}I)$ and $o(I) = \mathfrak{m}$ -adic order of $I = \max\{n \mid I \subseteq \mathfrak{m}^n\}$. Huneke and Sally [17, Theorem 2.1] proved that if R/\mathfrak{m} is infinite and $\mu(I) = 1 + o(I)$ then I is a contracted ideal. An important fact about complete ideal is that they are contracted [16, Proposition 3.1]. Let I be an \mathfrak{m} -primary ideal contracted from $S = R[\mathfrak{m}/x]$. Let N be a maximal ideal containing $\mathfrak{m}S$. Then S_N is a 2-dimensional regular local

ring. Suppose $o(I) = r$. Then $IS = x^r I' S$ for an ideal I' of S . We say I' is a transform of I in S and I'_N is called the transform of I in S_N . By Proposition 5 of [44, Appendix 5], if I is complete then so is I' and I'_N . Finally by [16, Proposition 3.6], $e(I) > e(I'_N)$.

Theorem 6.1. [41, Theorem 2.1] *Let (R, \mathfrak{m}) be a 2-dimensional regular local ring with R/\mathfrak{m} infinite. Let I and J be \mathfrak{m} -primary complete ideals. Then there exists a joint reduction (a, b) of (I, J) such that*

$$aJ + bI = IJ.$$

Proof. We may assume that R/\mathfrak{m} is infinite. Apply induction on $t = \max\{e(I), e(J)\}$. Let $t = 1$. Then $e(I) = 1 = \lambda(R/L)$ where L is a minimal reduction of I . Thus $I = L = \mathfrak{m}$ and similarly $J = \mathfrak{m}$. Since R is regular $\mathfrak{m} = (x, y)$ for some $x, y \in \mathfrak{m}$. Thus $x\mathfrak{m} + y\mathfrak{m} = \mathfrak{m}^2$. Now let $t > 1$. Let (a, b) be a joint reduction of (I, J) . We show that the ideal $K = aJ + bI$ is contracted. Let

$$aI^{n-1}J^n + bI^nJ^{n-1} = I^nJ^n$$

for some n . This equation implies that a (respectively b) is part of a minimal basis of I (respectively J). Let $o(I) = r$ and $o(J) = s$. Since I and J are complete, they are contracted. Hence $\mu(I) = r + 1$ and $\mu(J) = s + 1$. Let $I = (a_0, \dots, a_r)$ and $J = (b_0, \dots, b_s)$ where $a = a_0$ and $b = b_0$. Now we show that K is minimally generated by the $r + s + 1$ elements:

$$a_0b_0, a_0b_1, \dots, a_0b_s, b_0a_1, b_0a_2, \dots, b_0a_r.$$

Indeed, let

$$\sum_{i=0}^s a_0b_iu_i + \sum_{j=1}^r b_0a_jv_j = 0$$

where u_0, \dots, u_s and $v_1, \dots, v_r \in R$. As a, b is a regular sequence there is an $f \in R$ such that

$$a_0f = \sum_{j=1}^r a_jv_j \text{ and hence } b_0f = -\sum_{i=0}^s b_iu_i.$$

Hence $f, v_1, \dots, v_r \in \mathfrak{m}$ and $u_0 + f, u_1, \dots, u_s \in \mathfrak{m}$. Thus $\mu(K) = r + s + 1$. Since $K = aJ + bI$ is a reduction of IJ , $o(K) = o(IJ) = r + s$. Hence K is a contracted ideal. Pick $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that K and IJ are contracted from $S = R[\mathfrak{m}/x]$. We write $a = a'x^r$, $b = b'x^s$, $I = x^r I' S$, $J = x^s J' S$ where I', J' are ideals in S and $a', b' \in S$. Then

$$KS = (a'J' + b'I')x^{r+s}S, \quad IJS = x^{r+s}I'J'S.$$

Therefore (a', b') is a joint reduction of (I', J') . As IJ and K are contracted from S it is enough to show $a'J' + b'I' = I'J'$. To prove that $a'J' + b'I' = I'J'$, localize at any maximal ideal N of S containing $a'J' + b'I'$. Since $e(I'_N) < e(I)$ and $e(J'_N) < e(J)$, it follows that $(a'J' + b'I')S_N = I'J'S_N$. Therefore $a'J' + b'I' = I'J'$ and consequently $IJ = aJ + bI$.

Lemma 6.2. *Let (R, \mathfrak{m}) be a local ring of dimension ≥ 2 . Let I and J be ideals of R and $a \in I$, $b \in J$ be such that (a, b) is a regular sequence. Then the R -module homomorphism*

$$f : \frac{R}{I} \oplus \frac{R}{J} \longrightarrow \frac{(a, b)}{aJ + bI}$$

defined as $f(\overline{x}, \overline{y}) = (xb + ya) + (aJ + bI)$ is an isomorphism.

Proof. It is clear that f is surjective. For injectivity let $xb + ya \in aJ + bI$. Then $xb \in (a, bI)$. Choose $c \in R$ and $d \in I$ such that $xb = ac + bd$. Hence $b(x - d) = ca$ which implies $x - d \in (a : b) = (a)$. Thus $x \in I$. Similarly $y \in J$. Therefore f is injective and hence an isomorphism. \square

Theorem 6.3. *Let (R, \mathfrak{m}) be a 2-dimensional Cohen-Macaulay local ring with R/\mathfrak{m} infinite. Let I and J be \mathfrak{m} -primary ideals. Then the following are equivalent.*

- (a) *There exist $a \in I$ and $b \in J$ such that $aJ + bI = IJ$,*
- (b) *For all $r, s \geq 1$ $a^r J^s + b^s I^r = I^r J^s$,*
- (c) *For all $r, s \geq 1$, $\lambda(R/I^r J^s) = \lambda(R/I^r) + rse_1(I|J) + \lambda(R/J^s)$ where $e_1(I|J) = e(a, b)$,*
- (d) *$e_1(I|J) = \lambda(R/IJ) - \lambda(R/I) - \lambda(R/J)$.*

Proof. (a) \implies (b) By symmetry it is enough to show that $a^r J + bI^r = I^r J$ for all $r \geq 1$, Apply induction on r . Assume that $a^r J + bI^r = I^r J$. Then

$$\begin{aligned} I^{r+1}J &= I(I^r J) \\ &= I(a^r J + bI^r) \\ &= a^r IJ + bI^{r+1} \\ &= a^r(aJ + bI) + bI^{r+1} \\ &= a^{r+1}J + bI^{r+1}. \end{aligned}$$

(b) \implies (c) Since

$$R/I^r \oplus R/J^s \cong (a^r, b^s)/(a^r J^s + b^s I^r) = (a^r, b^s)/I^r J^s,$$

we get

$$\begin{aligned}
\lambda(R/I^r J^s) &= \lambda(R/(a^r, b^s)) + \lambda(R/I^r) + \lambda(R/J^s) \\
&= rs\lambda(R/(a, b)) + \lambda(R/I^r) + \lambda(R/J^s) \\
&= rse_1(I|J) + \lambda(R/I^r) + \lambda(R/J^s).
\end{aligned}$$

(c) \implies (d) Clear.

(d) \implies (a) As R/\mathfrak{m} is infinite there exists a joint reduction (a, b) of (I, J) . Since

$$R/I \oplus R/J \cong (a, b)/(aJ + bI),$$

we have $\lambda(R/aJ + bI) - e_1(I|J) = \lambda(R/I) + \lambda(R/J)$. Substitute $e_1(I|J) = \lambda(R/IJ) - \lambda(R/I) - \lambda(R/J)$ to get $IJ = aJ + bI$. \square

Corollary 6.4. [24] *Let I be an \mathfrak{m} -primary ideal of a regular local ring of dimension 2. Then for all $n \geq 1$,*

$$\lambda(R/\overline{I^n}) = e(I) \binom{n+1}{2} - [e(I) - \lambda(R/\overline{I})]n.$$

Proof. By Zariski's theorem, $\overline{I^n} = \overline{I}^n$ for all n . Hence we may assume that I is a complete ideal. Use induction on n . Assuming the result of $n-1$, we have

$$\begin{aligned}
\lambda(R/I^n) &= \lambda(R/I^{n-1}I) = \lambda(R/I^{n-1}) + (n-1)e(I) + \lambda(R/I) \\
&= e(I) \binom{n}{2} - [e(I) - \lambda(R/I)](n-1) + n - e(I) + \lambda(R/I) \\
&= e(I) \binom{n+1}{2} - [e(I) - \lambda(R/I)]n.
\end{aligned}$$

\square

Corollary 6.5. *Let I and J be \mathfrak{m} -primary ideals of a regular local ring of dimension 2. Then for all $r, s \geq 0$,*

$$\lambda(R/\overline{I^r J^s}) = e(I) \binom{r}{2} + rse_1(I|J) + e(J) \binom{s}{2} + r\lambda(R/\overline{I}) + s\lambda(R/\overline{J}).$$

Proof. We may assume that R/\mathfrak{m} is infinite. Thus for any joint reduction (a, b) of $(\overline{I}, \overline{J})$, $a\overline{J} + b\overline{I} = \overline{IJ}$. Thus

$$\lambda(R/\overline{I^r J^s}) = \lambda(R/\overline{I^r}) + rse_1(I|J) + \lambda(R/\overline{J^s}).$$

By Zariski's theorem $\overline{I^r} = \overline{I}^r$, $\overline{J^s} = \overline{J}^s$ and $\overline{I^r J^s} = \overline{I^r J^s}$. Now use Corollary 6.4 to finish the proof. \square

We end this section by sketching an alternate proof of Lipman's formula for normal Hilbert polynomial of an \mathfrak{m} -primary ideal in a regular local ring of dimension two. This is done by invoking a formula of Hoskin [12] which is also proved by Deligne [8] and Rees [35] independently. We refer the reader to section 14.5 of [17] for a very readable account.

Let I be an \mathfrak{m} -primary complete ideal in a 2-dimensional regular local ring (R, \mathfrak{m}) . We say that a 2-dimensional regular local ring (S, \mathfrak{n}) dominate R birationally if $R \subset S$, $\mathfrak{n} \cap R = \mathfrak{m}$ and R and S have equal fraction fields. Let N be a maximal ideal of $T = R[\mathfrak{m}/x]$ where $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathfrak{m}T \subset N$. Then T_N is a 2-dimensional regular local ring, called a local quadratic transform of R . Abhyankar [1, Theorem 3] showed that if S birationally dominates R then there is a unique sequence

$$R = R_0 \subset R_1 \subset \dots \subset R_n = S$$

of 2-dimensional regular local ring such that R_i is a local quadratic transform of R for $i = 1, \dots, n$. A point basis of I is the set

$$\mathcal{B}(I) = \{o(I^T) : T \text{ dominates } R \text{ birationally} \}$$

where I^T = transform of I in T and $o(I^T) = \mathfrak{m}_T$ -adic order of I^T . \square

Theorem 6.6 (Hoskin-Deligne Formula). *Let (R, \mathfrak{m}) be a regular local ring of dimension 2 with infinite residue field and let I be a complete \mathfrak{m} -primary ideal of R . Then*

$$\lambda(R/I) = \sum_{T \succ R} \binom{o(I^T) + 1}{2} [T/\mathfrak{m}_T : R/\mathfrak{m}]$$

where the sum is over all 2-dimensional regular local rings T which birationally dominate R , written as $T \succ R$ and \mathfrak{m}_T is the maximal ideal of T .

Corollary 6.7. *Let I be a complete \mathfrak{m} -primary ideal of 2-dimensional regular local ring (R, \mathfrak{m}) . Then for all $n \geq 1$,*

$$\lambda(R/I^n) = e(I) \binom{n+1}{2} - [e(I) - \lambda(R/I)]n.$$

Proof. By Zariski's theorem I^n is complete. Moreover $o((I^n)^T) = no(I^T)$. Hence for all $n \geq 1$,

$$\lambda(R/I^n) = \sum_{T \succ R} \binom{no(I^T) + 1}{2} [T/\mathfrak{m}_T : R/\mathfrak{m}].$$

Let us write $[T/\mathfrak{m}_T : R/\mathfrak{m}] = d(T)$ and $o(I^T) = o(T)$. Then for all $n \geq 0$,

$$\begin{aligned} \lambda(R/I^n) &= \sum_{T \succ R} \frac{1}{2} (n^2 o(T)^2 + n o(T)) d(T) \\ &= \frac{1}{2} e(I)(n^2 + n) - e_1(I)n + e_2(I) \end{aligned}$$

Hence

$$e(I) = \sum_{T \succ R} o(T)^2 d(T), \quad e_1(I) = \sum_{T \succ R} \binom{o(T)}{2} d(T), \quad e_2(I) = 0.$$

These expressions imply that $e_1(I) = e(I) - \lambda(R/I)$. \square

The Hoskin-Deligne formula has been generalized for finitely supported \mathfrak{m} -primary ideals in regular local rings of dimension at least three by C. D'Cruz [7]. B. Johnston [22] established a multiplicity formula for the same class of ideals.

7. THE NORMAL HILBERT POLYNOMIAL OF A MONOMIAL IDEAL

Let $R = k[x_1, x_2, \dots, x_d]$ be the polynomial ring over a field k with dimension $d \geq 2$. The maximal homogeneous ideal of R will be denoted by \mathfrak{m} . Let $I = (x^{v_1}, \dots, x^{v_q})$ be an \mathfrak{m} -primary monomial ideal where $v_i \in \mathbb{N}^d$ for $i = 1, 2, \dots, q$. If $w = (w_1, \dots, w_d) \in \mathbb{N}^d$ then we put $x^w = x_1^{w_1} \dots x_d^{w_d}$. First we describe the integral closure of I in terms of convex polytopes which will lead to a formula for the normal Hilbert polynomial of I in terms of Ehrhart polynomials of certain polytopes derived from the exponent vectors v_1, v_2, \dots, v_q .

Let e_1, e_2, \dots, e_d be the standard basis vectors of \mathbb{Q}^d . Since I is an \mathfrak{m} -primary ideal, there are natural numbers a_1, a_2, \dots, a_d such that $v_i = a_i e_i$ for $i = 1, 2, \dots, d$ after we have permuted the generators of I . Let $a = (1/a_1, 1/a_2, 1/a_3, \dots, 1/a_d)$. We may assume that $\langle v_i, a \rangle < 1$ for $i = d+1, \dots, s$ and $\langle v_i, a \rangle \geq 1$ for $i = s+1, \dots, q$. Consider the convex polytopes in \mathbb{Q}^d .

$$\begin{aligned} P &= \text{conv}(v_1, v_2, \dots, v_s) \\ S &= \text{conv}(0, v_1, v_2, \dots, v_d) \end{aligned}$$

and the convex polyhedron $Q = \mathbb{Q}_+^d + \text{conv}(v_1, v_2, \dots, v_q)$. B. Teissier [38] identified $\overline{I^n}$ in terms of lattice points of nQ . See also [43].

Proposition 7.1. *With above settings $\overline{I^n} = (\{x^a \mid a \in nQ \cap \mathbb{Z}^d\})$ for all $n \in \mathbb{N}$.*

Proof. Let $x^\alpha \in \overline{I^n}$ and $J = I^n$ then $x^\alpha \in \overline{J}$ which implies that there exists $0 \neq m \in \mathbb{N}$ such that $x^{m\alpha} \in J^m$. We have $I = (x^{v_1}, \dots, x^{v_q})$. Thus $x^{m\alpha} \in (x^{v_1}, \dots, x^{v_q})^{mn}$ which implies that $x^{m\alpha} = (x^{v_1})^{k_1} \dots (x^{v_q})^{k_q} x^\beta$ where $\sum k_i = mn$. Thus we get $m\alpha = \sum k_i v_i + \beta$ which implies $\alpha = \sum \frac{k_i}{m} v_i + \frac{\beta}{m}$. Let $l_i = \frac{k_i}{m}$ then $\alpha = \sum l_i v_i + \beta'$ where $\sum l_i = n$ and $0 \leq \beta' \in \mathbb{Q}^d$. Thus $\frac{\alpha}{n} = \sum \frac{l_i}{n} v_i + \frac{\beta'}{n}$ where $\sum \frac{l_i}{n} = 1$ which implies that $\frac{\alpha}{n} \in \text{conv}(v_1, \dots, v_q) + \mathbb{Q}_+^d = Q$. Therefore $\alpha \in nQ \cap \mathbb{Z}^d$.

Conversely let $\alpha \in nQ \cap \mathbb{Z}^d$ for some $n \geq 0$. Then $\alpha = u + w$ where $u \in n\text{conv}(v_1, \dots, v_q)$ and $w \in \mathbb{Q}_+^d$. Therefore $\alpha = n \sum l_i v_i + w$ with $0 \leq l_i \leq 1$ and $\sum l_i = 1$. Let $m = \text{l.c.m}(\text{denominators of } l_i)$ and $m_i = ml_i \in \mathbb{Z}$. Thus $m\alpha = mw + n \sum m_i v_i$ which implies that $x^{m\alpha} = x^{mw} ((x^{v_1})^{m_1} \dots (x^{v_q})^{m_q})^n$. Therefore $x^{m\alpha} \in I^{mn}$ and hence $x^\alpha \in \overline{I^n}$. \square

Villarreal [43] has shown that the normal Hilbert polynomial of a monomial ideal is the difference of two Ehrhart polynomials. First we recall the Ehrhart polynomial of a convex polytope and some of its properties.

Theorem 7.2 (Ehrhart, 1962). *Let P be an integral convex polytope of dimension d . Then the function $\chi_P(n) = |nP \cap \mathbb{Z}^d|$ for $n \in \mathbb{N}$ is a polynomial function of degree d denoted by*

$$E_P(n) = a_d n^d + \dots + a_1 n + a_0$$

with $a_i \in \mathbb{Q}$ for all i .

The polynomial $E_P(n)$ is called the **Ehrhart polynomial** of P . Some well known properties of E_P are

- (1) Let $\text{vol}(P)$ denote the relative volume of P . Then $a_d = \text{vol}(P)$.
- (2) Suppose F_1, \dots, F_s are facets of P . Then $a_{d-1} = (\sum_{i=1}^s \text{vol}(F_i)) / 2$.
- (3) We have $\chi_P(n) = E_P(n)$ for all integers $n \geq 0$.

Moralés first proved the following result in [29] later Villarreal [43] has given another proof.

Theorem 7.3. [43, Proposition 3.6] *Let I be an \mathfrak{m} -primary monomial ideal of the polynomial ring $k[x_1, x_2, \dots, x_d]$ over a field k . Then*

$$\lambda(R/\overline{I^n}) = |\mathbb{N}^d \setminus nQ| = E_S(n) - E_P(n)$$

for all n . In particular

$$\overline{P}_I(n) = [\text{vol}(S) - \text{vol}(P)]n^d + \text{lower degree terms}$$

and the constant term of $\overline{P}_I(x)$ is zero.

Proof. Since $\lambda(R/\overline{I^n}) = \dim_k(R/\overline{I^n})$, by Proposition 7.1, we have $\lambda(R/\overline{I^n}) = |\mathbb{N}^d \setminus nQ|$. As $E_S(0) = E_P(0) = 0$, we get the equality at $n = 0$. Now assume $n \geq 1$. Notice that we can decompose Q as $Q = (\mathbb{Q}_+^d \setminus S) \cup P$. Thus we get

$$nQ = (\mathbb{Q}_+^d \setminus nS) \cup nP \implies \mathbb{N}^d \setminus nQ = [\mathbb{N}^d \cap (nS)] \setminus [\mathbb{N}^d \cap (nP)].$$

Hence we get $\lambda(R/\overline{I^n}) = E_S(n) - E_P(n)$. Using the properties of Ehrhart polynomial we can write

$$\overline{P}_I(n) = [\text{vol}(S) - \text{vol}(P)]n^d + \text{lower degree terms}$$

with the constant term zero. \square

It follows from a result of Marley [27] that not only the normal Chern number but all the coefficients of the normal Hilbert polynomial of a monomial ideal in a polynomial ring are non-negative. We present a different proof of this theorem.

Theorem 7.4. *Let I be a zero-dimensional monomial ideal of a polynomial ring $R = k[x_1, x_2, \dots, x_d]$ over a field k . Then $\overline{e}_i(I) \geq 0$ for all $i = 0, 1, 2, \dots, d-1$.*

Proof. We may assume without loss of generality that k is infinite. The integral closure of a monomial ideal is a monomial ideal by [18, Proposition 1.4.6]. Hence The Rees algebra $\overline{\mathcal{R}} = \overline{\mathcal{R}}(I) = \bigoplus_{n=0}^{\infty} \overline{I^n} t^n$ is a normal semigroup ring. Hence by Hochster's theorem [4, Theorem 6.3.5] $\overline{\mathcal{R}}$ is Cohen-Macaulay. Therefore the associated graded ring $\overline{G} = \overline{G}(I) = \bigoplus_{n=0}^{\infty} \overline{I^n} / \overline{I^{n+1}}$ is Cohen-Macaulay by [42]. Let J be a minimal reduction of I . Then the initial forms of generators of J in degree one component of \overline{G} form a \overline{G} -regular sequence. Hence

$$\begin{aligned} H(\overline{G}/J\overline{G}, z) &= (1-z)^d H(\overline{G}, z) \\ &= \lambda(R/\overline{I}) + \lambda(\overline{I}/J + \overline{I^2})z + \dots + \lambda(\overline{I^{d-1}}/J\overline{I^{d-2}} + \overline{I^d})z^{d-1} \\ &:= f(z) \end{aligned}$$

In the above calculation we have used the celebrated theorem of Briançon-Skoda Theorem [18, Theorem 13.3.3] which asserts that $\overline{I^{n+d}} \subseteq I^{n+1}$ for all $n \geq 0$. Since for $i = 1, \dots, d-1$

$$i! \overline{e}_i(I) = \frac{d^i f(z)}{dz^i} \Big|_{z=1}.$$

Therefore $\overline{e}_i(I) \geq 0$ for $i = 0, 1, \dots, d-1$. \square

REFERENCES

- [1] S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. **78**(1956), 321-348.
- [2] C. Blancafort, *On Hilbert functions and cohomology*, J. Algebra **192** (1997), 439–459.
- [3] C. Blancafort, *Hilbert Functions: Combinatorial and Homological Aspects*, Ph. D. Thesis, (1997).
- [4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Revised Edition, Cambridge University Press, 1998.
- [5] C. Ciupercă, *Integral closure and generic elements*, J. Algebra **328** (2011), 122-131.
- [6] A. Corso, C. Polini and M. E. Rossi, *Depth of associated graded rings via Hilbert coefficients of ideals*, J. Pure Appl. Algebra **201** (2005), 126–141.
- [7] C. D' Cruz, *Integral closedness of MI and the formula of Hoskin and Deligne for finitely supported complete ideals* J. Algebra **304** (2006), 613632.
- [8] P. Deligne, *Intersections sur les surfaces régulières* (SGA 7, II), Lecture Notes in Mathematics, Vol 340, Springer-Verlag (1973), 1-38.
- [9] S. Goto, *Integral closedness of complete-intersection ideals*, J. Algebra **108** (1987), 151–160.
- [10] S. Goto, J. Hong and M. Mandal, *The positivity of the first coefficient of normal Hilbert polynomials*, Proc. Amer. Math. Soc. **139** (2011), 2399-2406.
- [11] J. Hong and B. Ulrich, *Specialization and integral closure*, Preprint (2008).
- [12] M. A. Hoskin, *Zero-dimensional valuation ideals associated with plane curve branches*, London Math. Soc. (3) **6** (1956), 70–99.
- [13] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. **56** (1997), 64–76.
- [14] S. Huckaba, and C. Huneke, *Normal ideals in regular rings*, J. Reine Angew. Math. **510** (1999), 63–82.
- [15] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318.
- [16] C. Huneke, *Complete ideals in two-dimensional regular local rings*, Commutative Algebra, Proceedings of a Microprogram, Springer-Verlag, (1989), 325-338.
- [17] C. Huneke and J. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra **115**(1988), 481-500.
- [18] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings and Modules*, London Mathematical Society Lecture Note Series **336**, Cambridge University Press, 2006.
- [19] S. Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), 390–401.
- [20] S. Itoh, *Coefficients of normal Hilbert polynomials*, J. Algebra **150** (1992), 101–117.
- [21] S. Itoh, *Hilbert Coefficients of Integrally Closed Ideals*, J. Algebra **176** (1995), 638–652.

- [22] B. Johnston, *The higher-dimensional multiplicity formula associated to the length formula of Hoskin and Deligne* Comm. Algebra **22** (1994), 2057-2071.
- [23] I. Kaplansky, *Commutative Rings*, Revised Edition, The University of Chicago Press, 1974.
- [24] J. Lipman, *On complete ideals in regular local rings*, Algebraic geometry and commutative algebra, Vol. I, Kinokuniya, Tokyo, (1988), 203-231.
- [25] J. Lipman and B. Teissier, *pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math J., **28** (1981), 97-116.
- [26] M. Mandal, B. Singh and J. K. Verma, *On some conjectures about the Chern numbers of filtrations*, Journal of Algebra **325** (2011), 147-162.
- [27] T. Marley, *Hilbert functions of ideals in Cohen-Macaulay rings*, Ph. D. Thesis, Purdue University, (1989).
- [28] M. Morálés, *Polynôme de Hilbert-Samuel des clôtures intégrales des puissances d'un idéal m -primaire*, Bulletin Soc. Math. France, **112** (1984), 343-358.
- [29] M. Morálés, *Polydre de Newton et genre gomtrique d'une singularit intersection complte*, Bulletin Soc. Math. France, **112** (1984), 325-341.
- [30] M. Morálés, and N. V. Trung, and O. Villamayor, *Sur la fonction de Hilbert-Samuel des clôtures intégrales des puissances d'idéaux engendrés par un système de paramètres*, J. Algebra **129** (1990), 96-102.
- [31] M. Nagata, *Local rings*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1975.
- [32] L. O'Carroll, *On two theorems concerning reductions in local rings*, J. Math. Kyoto Univ., **27** (1987), 61-67.
- [33] D. Rees, *A note on analytically unramified local rings*, J. London Math. Soc. **36** (1961), 24-28.
- [34] D. Rees, *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc, **29** (1984), 397-414.
- [35] D. Rees, *Hilbert functions and pseudo-rational local rings of dimension 2*, J. London Math. Soc, **24** (1981), 467-479.
- [36] M. E. Rossi and G. Valla, *Hilbert functions of filtered modules*, Lecture Notes of the Unione Matematica Italiana No. 9, Springer-Verlag, 2010.
- [37] J. D. Sally, *Ideals whose Hilbert function and Hilbert polynomial agree at $n = 1$* , J. Algebra **157** (1993), 534-547.
- [38] B. Teissier, *Monômes, volumes et multiplicités*. [Monomials, volumes and multiplicities] Introduction à la théorie des singularités, II, 127141, Travaux en Cours, 37, Hermann, Paris, 1988.
- [39] P. Valabrega and G. Valla, *Form rings and regular sequences*, Nagoya Math. J. **72** (1978), 93-101.
- [40] W. Vasconcelos, *The Chern coefficients of local rings*, Michigan Math. J. **57** (2008) 725-743.
- [41] J. K. Verma *Joint reduction of complete ideals*, Nagoya Math. J. **118** (1990), 155-163.
- [42] D. Q. Viet, *A note on the Cohen-Macaulayness of Rees algebras of filtrations*, Comm. Algebra **21** (1993), 221-229.

- [43] R. H. Villarreal, *Normalization of monomial ideals and Hilbert functions*, Proc. Amer. Math. Soc. **136** (2008), 1933–1943.
- [44] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. 2, van Nostrand, Princeton, 1960.

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